Waves in the Atmosphere and Oceans

Restoring Force

- ❑Conservation of potential temperature in the presence of positive static stability ➔ internal gravity waves
- ❑Conservation of potential vorticity in the presence of a mean gradient of potential vorticity ➔ Rossby waves

- **External gravity wave** (Shallow-water gravity wave)
- **Internal gravity (buoyancy) wave**
- **Inertial-gravity wave**: Gravity waves that have a large enough wavelength to be affected by the earth's rotation.
- **Rossby Wave**: Wavy motions results from the conservation of potential vorticity.
- **Kelvin wave**: It is a wave in the ocean or atmosphere that balances the Coriolis force against a topographic boundary such as a coastline, or a waveguide such as the equator. Kelvin wave is non-dispersive.

Lecture 6: Adjustment under Gravity in a Non-Rotating System

- Overview of Gravity waves
- Surface Gravity Waves
- "Shallow" Water
- Shallow-Water Model
- Dispersion

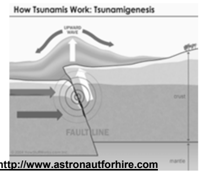
Chapter Five **Adjustment under Gravity in a Nonrotating System**

Goals of this Chapter

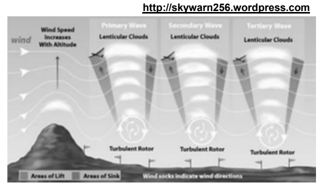
- This chapter marks the beginning of more detailed study of the way the atmosphere-ocean system *tends to adjust to equilibrium.*
- The adjustment processes are most easily understood in the absence of driving forces. Suppose, for instance, that the sun is "switched off," leaving the atmosphere and ocean with some non-equilibrium distribution of properties.
- How will they respond to the gravitational restoring force?
- Presumably there will be an adjustment to some sort of equilibrium. If so, what is the nature of the equilibrium?
- In this chapter, complications due to the rotation and shape of the earth will be ignored and only small departures from the hydrostatic equilibrium will be considered.
- The nature of the adjustment processes will be found by deduction from the equations of motion

Gravity Waves

http://skywarn256.wordpress.com



http://www.astronautforhire.com



- Gravity waves are waves generated in a fluid medium or at the interface between two media (e.g., the atmosphere and the ocean) which has the restoring force of gravity or buoyancy.
- When a fluid element is displaced on an interface or internally to a region with a different density, gravity tries to restore the parcel toward equilibrium resulting in an oscillation about the equilibrium state or wave orbit.
- Gravity waves on an air-sea interface are called surface gravity waves or surface waves while internal gravity waves are called internal waves.

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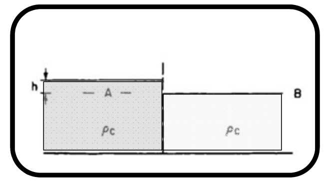
Adjustment Under Gravity in a Non-Rotating System

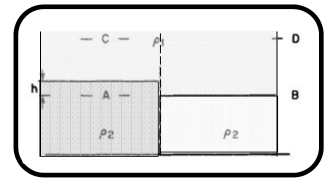
External Gravity Waves

adjustment of a homogeneous fluid with a free surface



Internal Gravity Waves

adjustment of a density-stratified fluid



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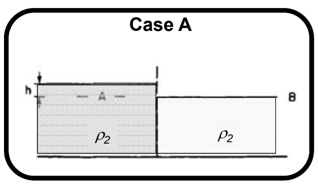
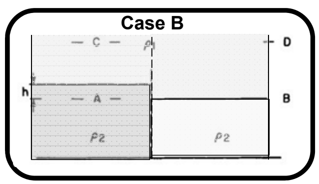
Reduced Gravity





Pressure difference between A and B:

$\Delta P = \rho_2 * g * h$

Pressure difference between A and B:

$\Delta P = (\rho_2 - \rho_1) * g * h$

buoyancy force = density difference * g

The adjustment process in Case B is exactly the same as in the Case A, except the gravitational acceleration is reduced to a value *g'*, where

$$g' = g(\rho_2 - \rho_1)/\rho_2.$$

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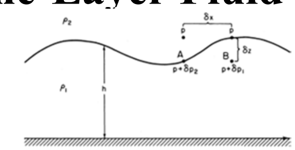
A One-Layer Fluid System



We assume that the motion is two dimensional in the x, z plane.

$\delta\rho = \rho_1 - \rho_2$.

Momentum Equation

$P_A = p + \delta p_1 = p + \rho_1 g \delta z = p + \rho_1 g (\partial h / \partial x) \delta x$

$P_B = p + \delta p_2 = p + \rho_2 g \delta z = p + \rho_2 g (\partial h / \partial x) \delta x$

❑ PG $= \lim_{\delta x \to 0} \left[\frac{(p + \delta p_1) - (p + \delta p_2)}{\delta x} \right] = g \delta\rho \frac{\partial h}{\partial x}$

❑ Momentum equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{g\delta\rho}{\rho_1}\frac{\partial h}{\partial x}$$

Continuity Equation

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

➔ $w(h) - w(0) = -h(\partial u / \partial x)$, where $w(h) = \frac{Dh}{Dt} = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x}$ and $w(0) = 0$

➔ $$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + h\frac{\partial u}{\partial x} = \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0$$

$$u = \overline{u} + u', \quad h = H + h'$$

applying the perturbation method

$$\frac{\partial u'}{\partial t} + \overline{u}\frac{\partial u'}{\partial x} + \frac{g\delta\rho}{\rho_1}\frac{\partial h'}{\partial x} = 0$$

$$\frac{\partial h'}{\partial t} + \overline{u}\frac{\partial h'}{\partial x} + H\frac{\partial u'}{\partial x} = 0$$

➔ $$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)^2 h' - \frac{gH\delta\rho}{\rho_1}\frac{\partial^2 h'}{\partial x^2} = 0$$

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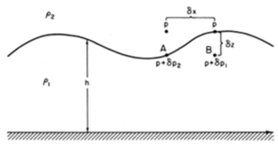
Shallow Water Gravity Wave



We assume that the motion is two dimensional in the x, z plane.

$\delta\rho = \rho_1 - \rho_2$

Governing Equation

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)^2 h' - \frac{gH\delta\rho}{\rho_1}\frac{\partial^2 h'}{\partial x^2} = 0$$

$$h' = A\exp[ik(x - ct)]$$

$$c = \overline{u} \pm \left(gH\delta\rho/\rho_1\right)^{1/2}$$

$\delta\rho \approx \rho_1$ (e.g., air and water)

$$c = \overline{u} \pm \sqrt{gH}$$

- The shallower the water, the slower the wave.
- Shallow water gravity waves are non-dispersive.

Shallow water wave speed ≈ 200 ms^{-1} for an ocean depth of 4km

- ❑ Shallow water gravity waves may also occur at thermocline where the surface water is separated from the deep water. *(These waves can also referred to as the internal gravity waves).*
- ❑ If the density changes by an amount $\delta\rho/\rho_1 \approx 0.01$, across the thermocline, then the wave speed for waves traveling along the thermocline will be only one-tenth of the surface wave speed for a fluid of the same depth.

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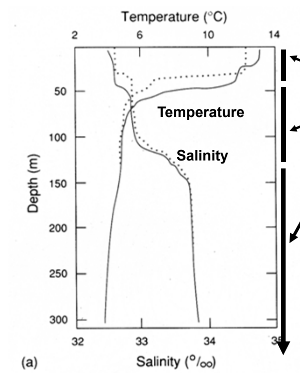
Vertical Structure of Ocean



Mixed Layer: T and S well mixed by winds

Thermocline: large gradient of T and S

Deep Ocean: T and S independent of height
cold
salty
high nutrient level

(from *Climate System Modeling*)

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Shallow and Deep Water

- "Shallow" in this lecture means that the depth of the fluid layer is small compared with the horizontal scale of the perturbation, i.e., the horizontal scale is large compared with the vertical scale.
- Shallow water gravity waves are the 'long wave approximation" end of gravity waves.
- Deep water gravity waves are the "short wave approximation" end of gravity waves.
- Deep water gravity waves are not important to large-scale motions in the oceans.

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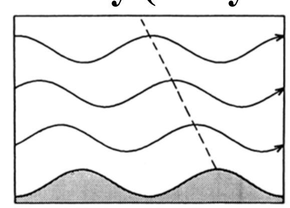
Internal Gravity (Buoyancy) Waves



- ❑ In a fluid, such as the ocean, which is bounded both above and below, gravity waves propagate primarily in the horizontal plane since vertically traveling waves are reflected from the boundaries to form standing waves.
- ❑ In a fluid that has no upper boundary, such as the atmosphere, gravity waves may propagate vertically as well as horizontally. In vertically propagating waves the phase is a function of height. Such waves are referred to as *internal waves.*
- ❑ *Although internal* gravity waves are not generally of great importance for synoptic-scale weather forecasting (and indeed are nonexistent in the filtered quasi-geostrophic models), they can be important in mesoscale motions.
- ❑ For example, they are responsible for the occurrence of mountain *lee waves. They also are believed to be an important* mechanism for transporting energy and momentum into the middle atmosphere, and are often associated with the formation of clear air turbulence (CAT).

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Quasi-Geostrophic Approximation

$$\frac{\partial \zeta_g}{\partial t} + \vec{V}_g \bullet \nabla \zeta_g + \beta v_g = -f \nabla \bullet \vec{V}$$

- Quasi-geostrophic approximation use the geostrophic wind for the actual wind everywhere *except* when computing divergence.
- The Q-G approximation eliminates both sound and gravity waves as solutions to the equations of motion.

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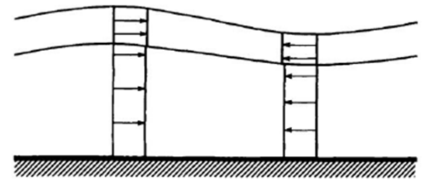
Lecture 7: Adjustment under Gravity of a Density-Stratified Fluid



- Normal Mode & Equivalent Depth
- Rigid Lid Approximation
- Boussinesq Approximation
- Buoyancy (Brunt-Väisälä) Frequency
- Dispersion of internal gravity waves

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Main Purpose of This Lecture

- As an introduction to the effects of stratification, the case of two superposed shallow layers, each of uniform density, is considered.
- In reality, both the atmosphere and ocean are continuously stratified.
- This serves to introduce the concepts of ***barotropic*** and ***baroclinic*** modes.
- This also serves to introduce two widely used approximations: the ***rigid lid approximation*** and the ***Boussinesq approximation***.

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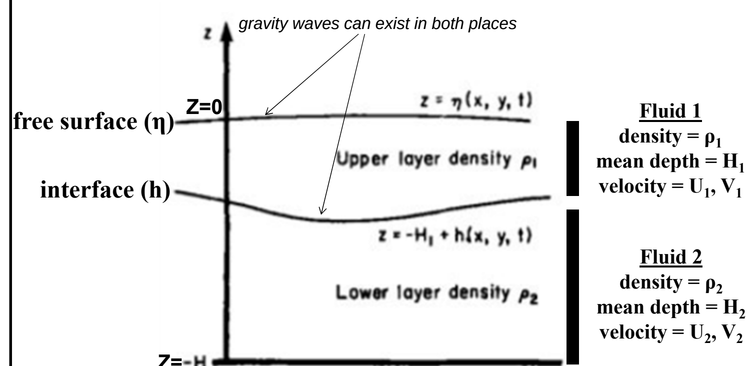
Two Fluids of Different Density



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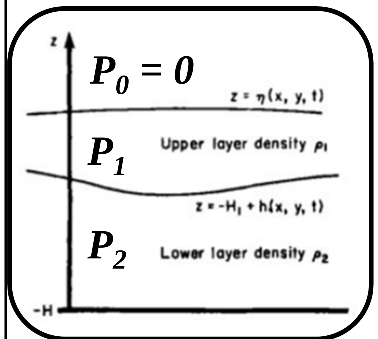
Two Fluids: *Layer 1* ($-H_1 + h < z < \eta$)



$$p_1 = \rho_1 g(\eta - z)$$

- ❑ **Momentum Equations**

$$\partial u_1/\partial t = -g\,\partial\eta/\partial x,$$
$$\partial v_1/\partial t = -g\,\partial\eta/\partial y,$$

- ❑ **Continuity Equation**

$$\partial(\eta + H_1 - h)/\partial t + H_1(\partial u_1/\partial x + \partial v_1/\partial y) = 0.$$

- ❑ **Taking time derivative of the continuity equation:**

$$\frac{\partial^2}{\partial t^2}(\eta - h) = H_1\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)g\eta \equiv gH_1\,\nabla^2\eta$$

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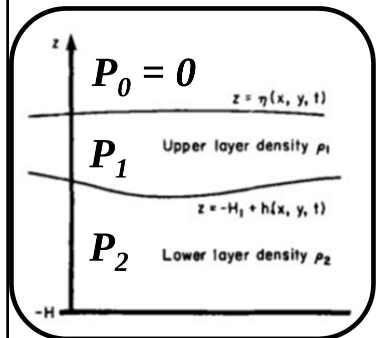
Two Fluids: *Layer 2* ($z < -H_1 + h$)

$$p_2 = \rho_1 g(\eta + H_1 - h) + \rho_2 g(-H_1 + h - z),$$



- ❑ **Momentum Equations**

$$\frac{\partial u_2}{\partial t} = -\frac{\rho_1}{\rho_2}g\frac{\partial\eta}{\partial x} - g'\frac{\partial h}{\partial x}$$
$$\frac{\partial v_2}{\partial t} = -\frac{\rho_1}{\rho_2}g\frac{\partial\eta}{\partial y} - g'\frac{\partial h}{\partial y}$$
$$g' = g(\rho_2 - \rho_1)/\rho_2$$

= reduced gravity

- ❑ **Continuity Equation**

$$\partial h/\partial t + H_2(\partial u_2/\partial x + \partial v_2/\partial y) = 0.$$

- ❑ **Taking time derivative of the continuity**

$$\frac{\partial^2 h}{\partial t^2} = H_2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\rho_1}{\rho_2}g\eta + g'h\right) = H_2\,\nabla^2(g\eta - g'\eta + g'h),$$

Adjustments of the Two-Fluid System

- ❑ **The adjustments in the two-layer fluid system are governed by:**

$$\frac{\partial^2}{\partial t^2}(\eta - h) = H_1\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)g\eta \equiv gH_1\,\nabla^2\eta$$
$$\frac{\partial^2 h}{\partial t^2} = H_2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\rho_1}{\rho_2}g\eta + g'h\right) = H_2\,\nabla^2(g\eta - g'\eta + g'h),$$

- ❑ **Combined these two equations will result in a fourth-order equation, which is difficult to solve.**
- ❑ **This problem can be greatly simplified by looking for solutions which η and h are proportional:**

$$h(x, y, t) = \mu\eta(x, y, t),$$

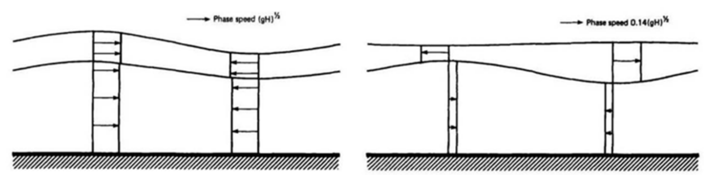
- ❑ **The governing equations will both reduced to this form:**

$$\partial^2\eta/\partial t^2 = c_e^2\,\nabla^2\eta,$$ provided that

$$gH_1/(1 - \mu) = \mu^{-1}(g - g'(1 - \mu))H_2 = c_e^2.$$

There are two values of μ (and hence two values of c_e) that satisfy this equation.
➔ The motions corresponding to these particular vales are called normal modes of oscillation.

Normal Modes



barotropic mode **baroclinic mode**

- The motions corresponding to these particular values of c_e or μ are called ***normal modes*** of oscillation.
- In a system consisting of n layers of different density, there are n normal modes corresponding to the n degrees of freedom.
- A continuously stratified fluid corresponding to an infinite number of layers, and so there is an infinite set of modes.

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Structures of the Normal Modes

- The structures of the normal modes can be obtained by solving this equation (from previous slide):

$$c_e^4 - gHc_e^2 + gg'H_1H_2 = 0,$$
$$H = H_1 + H_2$$

- Or, we can use the concept of the one-layer shallow water model, where the phase speed (c) of the gravity wave is related to the depth of the shallow water (H):

$$c = \sqrt{gH}$$

- Using this concept, we can assume each of the normal mode behaves like the one-layer shallow water with a "equivalent depth" of H_e:

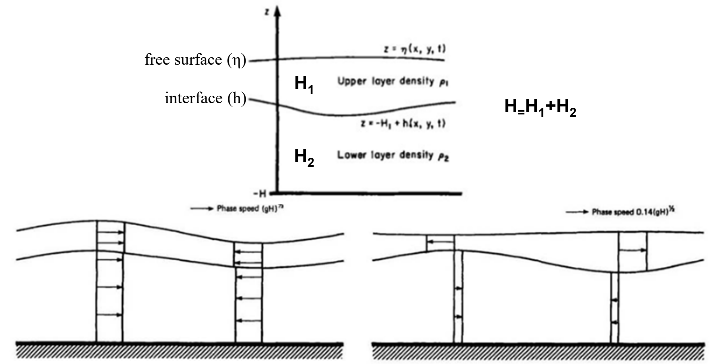
$$c_e^2 = gH_e.$$

$$gH_e^2 - gHH_e + g'H_1H_2 = 0.$$

Solution 1 (brotropic mode):
$c_0^2 = gH(1 - g'H_1H_2/gH^2 \cdots)$,
$\eta/h \approx H/H_2$, $u_2/u_1 = 1 - g'H_1/gH \cdots$

Solution 2 (baroclinic mode):
$c_1^2 = (g'H_1H_2/H)(1 + g'H_1H_2/gH^2 \cdots)$,
$\eta/h \approx -g'H_2/gH$, $u_2/u_1 \approx -H_1/H_2$.

Structures of the Normal Modes



Barotropic mode

$c_0^2 = gH(1 - g'H_1H_2/gH^2 \cdots)$,
$\eta/h \approx H/H_2$, $u_2/u_1 = 1 - g'H_1/gH \cdots$

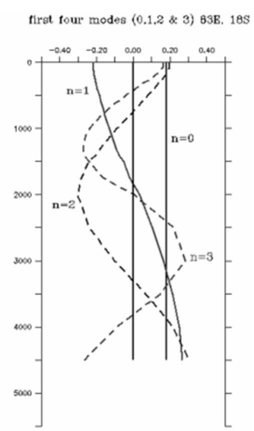
$\eta > h$; u_2&u_1 of same signs

Baroclinic mode

$c_1^2 = (g'H_1H_2/H)(1 + g'H_1H_2/gH^2 \cdots)$,
$\eta/h \approx -g'H_2/gH$, $u_2/u_1 \approx -H_1/H_2$.

$\eta < h$; u_2&u_1 of opposite signs

Example



Equivalent Depth (H_e)

- An N-layer fluid will have one barotropic mode and (N-1) baroclinic modes of gravity waves, each of which has its own equivalent depth.
- Once the equivalent depth is known, we know the dispersion relation of that mode of gravity wave and we know how fast/slow that gravity wave propagates.

$$c_e^2 = gH_e.$$

For the barotropic mode: He=H
For the baroclibnic mode: He = g'/g * H_1H_2/H

- For a continuously stratified fluid, it has an infinite number of modes, but not all the modes are important. We only need to identify the major baroclinic modes and to find out their equivalent depths.

An example of Equivalent Atmosphere

- Equivalent depth for an incompressible atmosphere

$$h_0 = \frac{N^2 H^2}{\pi^2 g}.$$

- Rossby radius of deformation

For a barotropic ocean: $L_R \equiv \frac{(gD)^{1/2}}{f}$

The *n*th baroclinic Rossby radius is:

$L_{R,n} \equiv \frac{NH}{n\pi f_0}$, where N is the Brunt–Väisälä frequency, H is the scale height, and n = 1, 2, ...

- The **gravity wave speed, and thus the Rossby radius**, increases proportionally with the depth of the disturbance.
- The gravity wave speed, and thus the Rossby radius, increases with stability by around a factor of two from steep lapse rates to isothermal conditions.

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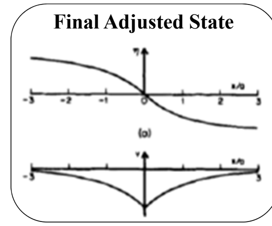
Final Adjusted State



Radius of deformation:

$a = c/|f| = (gH)^{1/2}/|f|$

$$\frac{\eta}{\eta_0} = \begin{cases} -1 + e^{-x/a} & \text{for } x > 0 \\ 1 - e^{x/a} & \text{for } x < 0, \end{cases}$$

$$v = -(g\eta_0/fa)\exp(-|x|/a).$$

- The steady equilibrium solution is *not one of rest, but is a geostrophic balance.*
- The equation determining this steady solution contains a length scale *a,* called the Rossby radius of deformation.
- The energy analysis indicates that *energy is hard to extract from a rotating* fluid. In the problem studied, there was an infinite amount of potential energy available for conversion into kinetic energy, but only a finite amount of this available energy was released. The reason was that a geostrophic equilibrium was established, and such an equilibrium retains potential energy.

Rossby Radius of Deformation

For Barotropic Flow	For Baroclinic Flow
$L_R \equiv \frac{(gD)^{1/2}}{f_0}$, (water depth)	$L_R \equiv \frac{NH}{f_0}$, (Brunt-Vaisala frequency; equivalent depth)

- In atmospheric dynamics and physical oceanography, the Rossby radius of deformation is the length scale at which *rotational effects* become as important as *buoyancy or gravity wave effects* in the evolution of the flow about some disturbance.
- "deformation": It is the radius that the direction of the flow will be "deformed" by the Coriolis force from straight down the pressure gradient to be in parallel to the isobars.
- The size of the radius depends on the stratification (how density or potential temperature changes with height) and Coriolis parameter.
- The Rossby radius is considerably larger near the equator.

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Rossby Number

$$R_0 \equiv U/(f_0 L)$$

- Rossby number is a non-dimensional measure of the magnitude of the acceleration compared to the Coriolis force:

$$(U^2/L)/(f_0 U)$$

- The smaller the Rossby number, the better the geostrophic balance can be used.

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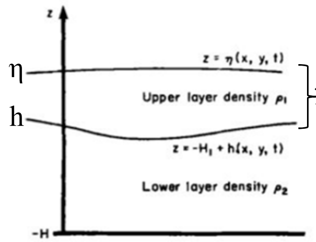
Rigid Lid Approximation
(for the upper layer)



$$p_1 = \rho_1 g(\eta - z)$$

❑ **Momentum Equations**

$$\partial u_1/\partial t = -g\,\partial\eta/\partial x,$$
$$\partial v_1/\partial t = -g\,\partial\eta/\partial y,$$

❑ **Continuity Equation**

$$\partial(\eta + H_1 - h)/\partial t + H_1(\partial u_1/\partial x + \partial v_1/\partial y) = 0.$$

- For baroclinic modes, surface displacements (η) are small compared to interface displacements (h).
- If there is a rigid lid at z=0, the identical pressure gradients would have be achieved.

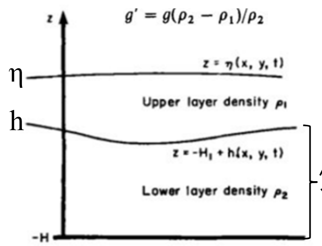
$$-\partial h/\partial t + H_1(\partial u_1/\partial x + \partial v_1/\partial y) = 0.$$

Purpose of Rigid Lid Approximation

- Rigid lid approximation: the upper surface was held fixed but could support pressure changes related to waves of lower speed and currents of interest.
- Ocean models us***ed*** the "rigid lid" approximation to eliminate high-speed external gravity waves and allow a longer time step.
- As a result, ocean tides and other waves having the speed of tsunamis were filtered out.
- The **rigid lid approximation** was used in the 70's to filter out gravity wave dynamics in ocean models. Since then, ocean model have evolved to include a free-surface allowing fast-moving gravity wave physics.

Boussinesq Approximation
(for the lower layer)

$$g' = g(\rho_2 - \rho_1)/\rho_2$$



❑ **Momentum Equations**

$$\frac{\partial u_2}{\partial t} = -\frac{\rho_1}{\rho_2} g \frac{\partial\eta}{\partial x} - g' \frac{\partial h}{\partial x}$$
$$\frac{\partial v_2}{\partial t} = -\frac{\rho_1}{\rho_2} g \frac{\partial\eta}{\partial y} - g' \frac{\partial h}{\partial y}$$

❑ **Continuity Equation**

$$\partial h/\partial t + H_2(\partial u_2/\partial x + \partial v_2/\partial y) = 0.$$

- Boussinesq approx: replace the ratio (ρ_1/ρ_2) by unity in the momentum equation.

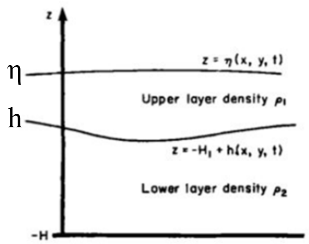
$$\partial u_2/\partial t = -g\,\partial\eta/\partial x - g'\,\partial h/\partial x,$$
$$\partial v_2/\partial t = -g\,\partial\eta/\partial y - g'\,\partial h/\partial y.$$

- We keep the density difference in this g′ term, because it involves density difference (ρ_1- ρ_2)/ ρ_1*g, which is related to the buoyancy force.

Purpose of Boussinesq Approximation

- This approximation states that density differences are sufficiently small to be neglected, except where they appear in terms multiplied by *g*, the acceleration due to gravity (i.e., buoyancy).
- In the Boussinesq approximation, which is appropriate for an almost-incompressible fluid, it assumed that variations of density are small, so that in the inertial terms, and in the continuity equation, we may substitute ρ by ρ_0, a constant. However, even weak density variations are important in buoyancy, and so we retain variations in ρ in the buoyancy term in the vertical equation of motion.
- *Sound waves are impossible/neglected when the Boussinesq approximation is used, because sound waves move via density variations.*
- Boussinesq approximation is for the problems that the variations of temperature as well as the variations of density are small. In these cases, the variations in volume expansion due to temperature gradients will also small. For these case, Boussinesq approximation can simplify the problems and save computational time.

After Using the Two Approximations



❑ **Upper layer**

$$\partial u_1/\partial t = -g\,\partial\eta/\partial x,$$
$$\partial v_1/\partial t = -g\,\partial\eta/\partial y,$$
$$-\partial h/\partial t + H_1(\partial u_1/\partial x + \partial v_1/\partial y) = 0.$$

❑ **Lower layer**

$$\partial u_2/\partial t = -g\,\partial\eta/\partial x - g'\,\partial h/\partial x,$$
$$\partial v_2/\partial t = -g\,\partial\eta/\partial y - g'\,\partial h/\partial y.$$
$$\partial h/\partial t + H_2(\partial u_2/\partial x + \partial v_2/\partial y) = 0.$$

- After the approximations, there is no η in the two continuity equation ➔ They can be combined to become one equation.
- The two momentum equations can also be combined into one single equation without η.
- At the end, the continuity and momentum equations for the upper and lower layers can be combined to solve for the dispersive relation for the baroclinic mode of the gravity wave.

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Brunt–Väisälä Frequency (*N*)

❑ Consider a parcel of (water or gas) that has density of ρ_0 and the environment with a density that is a function of height: $\rho = \rho(z)$. If the parcel is displaced by a small vertical increment z', it will subject to an extra gravitational force against its surroundings of:

$$\rho_0 \frac{\partial^2 z'}{\partial t^2} = -g(\rho_0 - \rho(z'))$$

$$\rho(z) - \rho_0 = \frac{\partial \rho(z)}{\partial z} z',$$

➔ $$\frac{\partial^2 z'}{\partial t^2} = \frac{g}{\rho_0}\frac{\partial \rho(z)}{\partial z} z'$$

➔ $$z' = z'_0 e^{\sqrt{-N^2}t}$$

where $N = \sqrt{-\frac{g}{\rho_0}\frac{\partial \rho(z)}{\partial z}}$ *for oceans*, $N \equiv \sqrt{\frac{g}{\theta}\frac{d\theta}{dz}},$ *for the atmosphere*

- A fluid parcel in the presence of stable stratification ($N^2 > 0$) will oscillate vertically if perturbed vertically from its starting position.
- In atmospheric dynamics, oceanography, and geophysics, the Brunt-Vaisala frequency, or buoyancy frequency, is the angular frequency at which a vertically displaced parcel will oscillate within statically stable environment.
- The Brunt–Väisälä frequency relates to internal gravity waves and provides a useful description of atmospheric and oceanic stability.

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Internal Gravity Waves in Atmosphere and Oceans

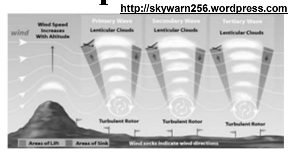
In Oceans

$$N = \sqrt{-\frac{g}{\rho_0}\frac{\partial \rho(z)}{\partial z}}$$

In Atmosphere

$$N \equiv \sqrt{\frac{g}{\theta}\frac{d\theta}{dz}},$$

http://skywarn256.wordpress.com



❑ Internal gravity waves can be found in both the ***statically stable*** (dΘ/dz>0) atmosphere and the ***stably stratified*** (-dρ/dz>0)ocean.

❑ The buoyancy frequency for the internal gravity wave in the ocean is determined by the vertical density gradient, while it is determined by the vertical gradient of potential temperature in the atmosphere.

❑ In the troposphere, the typical value of N is 0.01 sec^{-1}, which correspond to a period of about 10 minutes.

❑ Although there are plenty of gravity waves in the atmosphere, most of them have small amplitudes in the troposphere and are not important, except that the gravity waves generated by flows over mountains. These mountain waves can have large amplitudes.

❑ Gravity waves become more important when they propagate into the upper atmosphere (particularly in the mesosphere) where their amplitudes got amplified due to low air density there.

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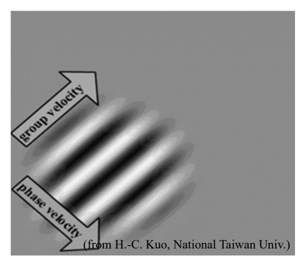
Dispersion of Internal Gravity Waves

mean flow, zonal wavenumber, vertical wavenumber, total wavenumber

$$\hat{\nu} \equiv \nu - \overline{u}k = \pm Nk/\left(k^2 + m^2\right)^{1/2} = \pm Nk/|\boldsymbol{\kappa}|$$

$\hat{\nu}$ ***is always smaller than N!!***

Internal gravity waves can have any frequency between zero and a maximum value of N.



(from H.-C. Kuo, National Taiwan Univ.)

❑ In the atmosphere, internal gravity waves generated in the troposphere by cumulus convection, by flow over topography, and by other processes may propagate upward many scale heights into the middle atmosphere.

❑ Phase velocity:

$$c_x = \hat{\nu}/k \text{ and }, c_z = \hat{\nu}/m$$

❑ Group velocity:

$$c_{gx} = \frac{\partial \nu}{\partial k} = \overline{u} \pm \frac{Nm^2}{\left(k^2 + m^2\right)^{3/2}}$$

$$c_{gz} = \frac{\partial \nu}{\partial m} = \pm \frac{(-Nkm)}{\left(k^2 + m^2\right)^{3/2}}$$

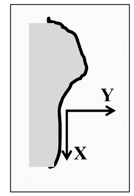
❑ The phase velocity and group velocity are perpendicular and that the vertical components of the phase and group velocities have opposite sign: if a wavepacket moves upward to the right, the crests move downward to the right.

Kelvin Waves

- The Kelvin wave is a large-scale wave motion of great practical importance in the Earth's atmosphere and ocean.
- The Kelvin wave is a special type of gravity wave that is affected by the Earth's rotation and trapped at the Equator or along lateral vertical boundaries such as coastlines or mountain ranges.
- The existence of the Kelvin wave relies on (a) gravity and stable stratification for sustaining a gravitational oscillation, (b) significant Coriolis acceleration, and (c) the presence of vertical boundaries or the equator.
- There are two basic types of Kelvin waves: boundary trapped and equatorially trapped. Each type of Kelvin wave may be further subdivided into surface and internal Kelvin waves.
- Atmospheric Kelvin waves play an important role in the adjustment of the tropical atmosphere to convective latent heat release, in the stratospheric quasi-biennial oscillation, and in the generation and maintenance of the Madden–Julian Oscillation.
- Oceanic Kelvin waves play a critical role in tidal motion, in the adjustment of the tropical ocean to wind stress forcing, and in generating and sustaining the El Nino Southern Oscillation.

(from Bin Wang 2002)

Kelvin Waves



Governing Equations

$$\frac{du}{dt} - fv = -g\frac{\partial h}{\partial x};$$

$$\frac{dv}{dt} + fu = -g\frac{dh}{dy};$$

$$\frac{dh}{dt} + D\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0.$$

A unique boundary condition

$y = 0$ is $v = 0$

$$\begin{pmatrix} u' \\ h' \end{pmatrix} = \mathrm{Re}\left\{\begin{pmatrix} U(y) \\ H(y) \end{pmatrix} \exp\left[ik(x-ct)\right]\right\}$$

$$H = const \times \exp\left(-\frac{f}{c}y\right)$$

$$-\frac{f}{g}U = -\frac{f}{c}H$$

$$c = \sqrt{gD}$$

- A Kelvin wave is a type of low-frequency gravity wave in the ocean or atmosphere that balances the Earth's Coriolis force against a topographic boundary such as a coastline, or a waveguide such as the equator.
- Therefore, there are two types of Kelvin waves: coastal and equatorial.
- A feature of a Kelvin wave is that it is non-dispersive, i.e., the phase speed of the wave crests is equal to the group speed of the wave energy for all frequencies.

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Costal Kelvin Waves

$$H = const \times \exp\left(-\frac{f}{c}y\right)$$

At the coast $y = 0$ is $v = 0$:

$$c = \sqrt{gD}$$

depth of the fluid

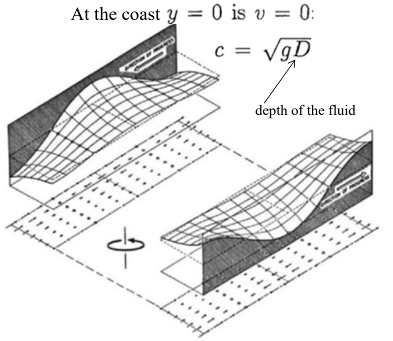

Fig. 10.3. Northern hemisphere Kelvin waves on opposite sides of a channel that is wide compared with the Rossby radius. In each vertical plane parallel to the coast, the currents (shown by arrows) are entirely within the plane and are exactly the same as those for a long gravity wave in a nonrotating channel. However, the surface elevation varies exponentially with distance from the coast in order to give a geostrophic balance. This means Kelvin waves move with the coast on their right in the northern hemisphere and on their left in the southern hemi-

- *Coastal Kelvin waves always propagate with the shoreline* on the right in the northern hemisphere and on the left in the southern hemisphere.
- In each vertical plane to the coast, the currents (shown by arrows) *are entirely within the plane* and are exactly the same as those for a long gravity wave in a non-rotating channel.
- However, the surface elevation varies exponentially with distance from the coast in order to give a geostrophic balance.

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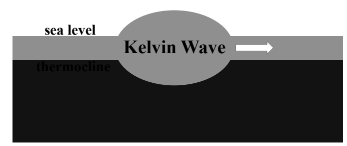
Equatorial Kelvin Waves



- The equator acts analogously to a topographic boundary for both the Northern and Southern Hemispheres, which make the equatorial Kelvin wave to behaves very similar to the coastally-trapped Kelvin wave.
- Surface equatorial Kelvin waves travel very fast, at about 200 m per second. Kelvin waves in the thermocline are however much slower, typically between 0.5 and 3.0 m per second.
- They may be detectable at the surface, as sea-level is slightly raised above regions where the thermocline is depressed and slightly depressed above regions where the thermocline is raised.
- The amplitude of the Kelvin wave is several tens of meters along the thermocline, and the length of the wave is thousands of kilometres.
- Equatorial Kelvin waves can only travel eastwards.

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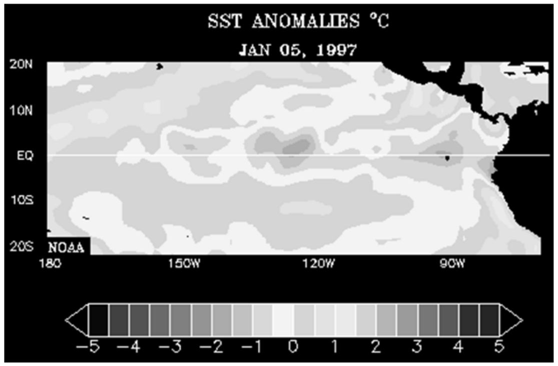
1997-98 El Nino



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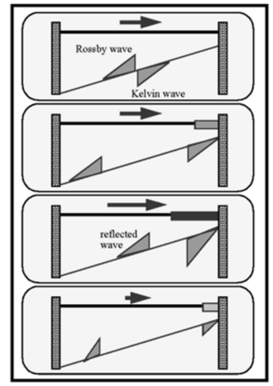
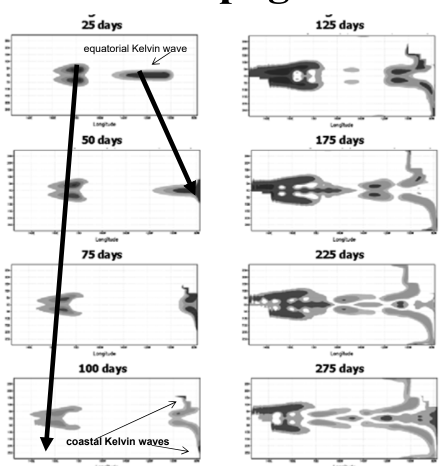
Delayed Oscillator Theory



- Wind forcing at the central Pacific: produces a downwelling Kevin wave propagating eastward and a upwelling Rossby wave propagating westward.
- wave propagation: the fast kelvin wave causes SST warming at the eastern basin, while slow Rossby wave is reflected at the western boundary.
- wave reflection: Rossby wave is reflected as a upwelling Kelvin wave and propagates back to the eastern basin to reverse the phase of the ENSO cycle.
- ENSO period: is determined by the propagation time of the waves.

Wave Propagation and Reflection



(Figures from IRI)

- It takes Kevin wave (phase speed = 2.9 m/s for the first baroclinic mode) about 70 days to cross the Pacific basin (17,760km).
- It takes Rossby wave about 200 days (phase speed = 0.93 m/s) to cross the Pacific basin.

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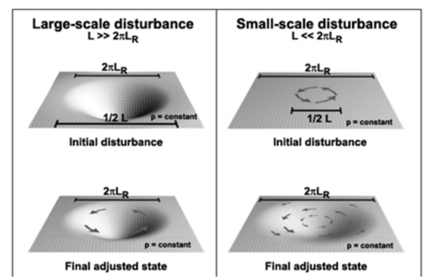
Lecture 8: Adjustment in a Rotating System



- Geostrophic Adjustment Process
- Rossby Radius of Deformation

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Geostrophic Adjustments

- The atmosphere is nearly always close to geostrophic and hydrostatic balance.
- If this balance is disturbed through such processes as heating or cooling, the atmosphere adjusts itself to get back into balance. This process is called *geostrophic adjustment.*
- A key feature in the geostrophic adjustment process is that pressure and velocity fields have to adjust to each other in order to reach a geostrophic balance. When the balance is achieved, the flow at any level is along the isobars.
- We can study the geostrophic adjustment by studying the adjustment in a barotropic fluid using the shallow-water equations.
- The results can be extended to a baroclinic fluid by using the concept of equivalent depth.

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Geostrophic Adjustment Problem

shallow water model

$$\frac{\partial u'}{\partial t} - f_0 v' = -g\frac{\partial h'}{\partial x}$$

$$\frac{\partial v'}{\partial t} + f_0 u' = -g\frac{\partial h'}{\partial y}$$

$$\frac{\partial h'}{\partial t} + H\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) = 0$$

$$\Rightarrow \quad \frac{\partial^2 h'}{\partial t^2} - c^2\left(\frac{\partial^2 h'}{\partial x^2} + \frac{\partial^2 h'}{\partial y^2}\right) + f_0 H\zeta' = 0$$

$$\frac{\partial \zeta'}{\partial t} + f_0\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) = 0 \Rightarrow \frac{\partial \zeta'}{\partial t} - \frac{f_0}{H}\frac{\partial h'}{\partial t} = 0 \Rightarrow Q'(x, y, t) = \zeta'/f_0 - h'/H = \text{Const.}$$

If we know the distribution of perturbation potential vorticity (Q') at the initial time, we know for all time: $Q'(x, y, t) = Q'(x, y, 0)$

And the <u>final adjusted state</u> can be determined without solving the time-dependent problem.

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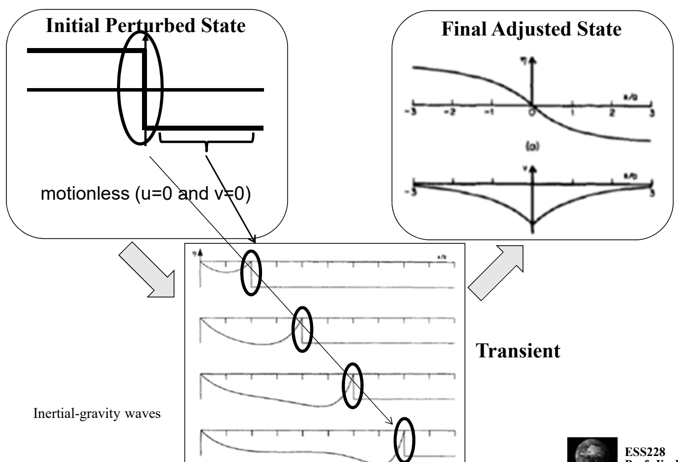
An Example of Geostrophic Adjustment



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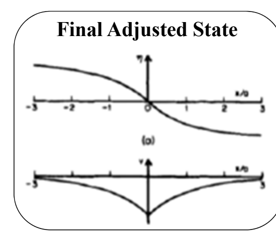
Final Adjusted State



Radius of deformation:

$$a = c/|f| = (gH)^{1/2}/|f|$$

$$\frac{\eta}{\eta_0} = \begin{cases} -1 + e^{-x/a} & \text{for } x > 0 \\ 1 - e^{x/a} & \text{for } x < 0, \end{cases}$$

$$v = -(g\eta_0/fa)\exp(-|x|/a).$$

- The steady equilibrium solution is *not one of rest, but is a geostrophic balance.*
- The equation determining this steady solution contains a length scale *a*, called the Rossby radius of deformation.
- The energy analysis indicates that *energy is hard to extract from a rotating* fluid. In the problem studied, there was an infinite amount of potential energy available for conversion into kinetic energy, but only a finite amount of this available energy was released. The reason was that a geostrophic equilibrium was established, and such an equilibrium retains potential energy.

Rossby Radius of Deformation

For Barotropic Flow

$$L_R \equiv \frac{(gD)^{1/2}}{f_0},$$

water depth

For Baroclinic Flow

Brunt-Vaisala frequency

$$L_R \equiv \frac{NH}{f_0},$$

equivalent depth

- In atmospheric dynamics and physical oceanography, the Rossby radius of deformation is the length scale at which *rotational effects* become as important as *buoyancy or gravity wave effects* in the evolution of the flow about some disturbance.
- "deformation": It is the radius that the direction of the flow will be "deformed" by the Coriolis force from straight down the pressure gradient to be in parallel to the isobars.
- The size of the radius depends on the stratification (how density or potential temperature changes with height) and Coriolis parameter.
- The Rossby radius is considerably larger near the equator.

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Rossby Radius and the Equilibrium State

Mass and Velocity

$$-\frac{\zeta}{f} : \frac{\eta}{H} = \kappa_H^2 a^2 : 1$$

wave number deformation radius

Energy Partition

$$\text{K.E.} : \text{P.E.} = \kappa_H^2 a^2 : 1,$$

- For large scales ($K_H a \ll 1$), the potential vorticity perturbation is mainly associated with perturbations in the mass field, and that the energy changes are in the potential and internal forms.
- For small scales ($K_H a \gg 1$) potential vorticity perturbations are associated with the velocity field, and the energy perturbation is mainly kinetic.
- At large scales ($K_H^{-1} \gg$ *a; or* $K_H a \ll 1$), it is the mass field that is determined by the initial potential vorticity, and the velocity field is merely that which is in geostrophic equilibrium with the mass field. It is said, therefore, that the large-scale velocity field adjusts to be in equilibrium with the large scale mass field.
- At small scales ($K_H^{-1} \ll$ *a) it is the velocity field* that is determined by the initial potential vorticity, and the mass field is merely that which is in geostrophic equilibrium with the velocity field. In this case it can be said that the mass field adjusts to be in equilibrium with the velocity field.

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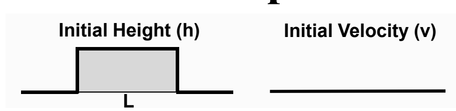
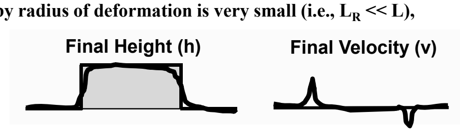
Examples

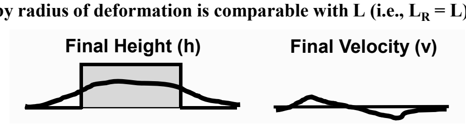


- **If the Rossby radius of deformation is very small (i.e., $L_R << L$),**



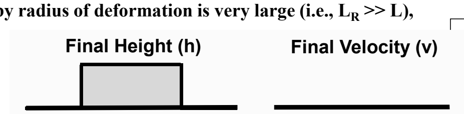
velocity adjusted to mass

- **If the Rossby radius of deformation is comparable with L (i.e., $L_R = L$),**



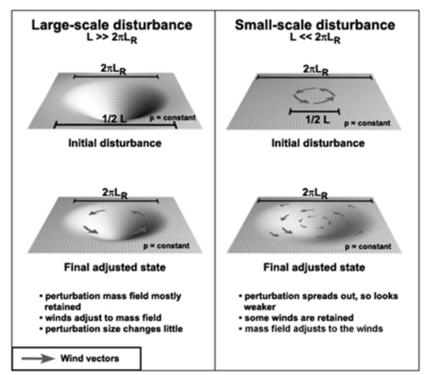
velocity and mass both adjusted

- **If the Rossby radius of deformation is very large (i.e., $L_R >> L$),**



mass adjusted to velocity

Rossby Radius and the Equilibrium State



- If the size of the disturbance is much larger than the Rossby radius of deformation, then the velocity field adjusts to the initial mass (height) field.
- If the size of the disturbance is much smaller than the Rossby radius of deformation, then the mass field adjusts to the initial velocity field.
- If the size of the disturbance is close to the Rossby radius of deformation, then both the velocity and mass fields undergo mutual adjustment.

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