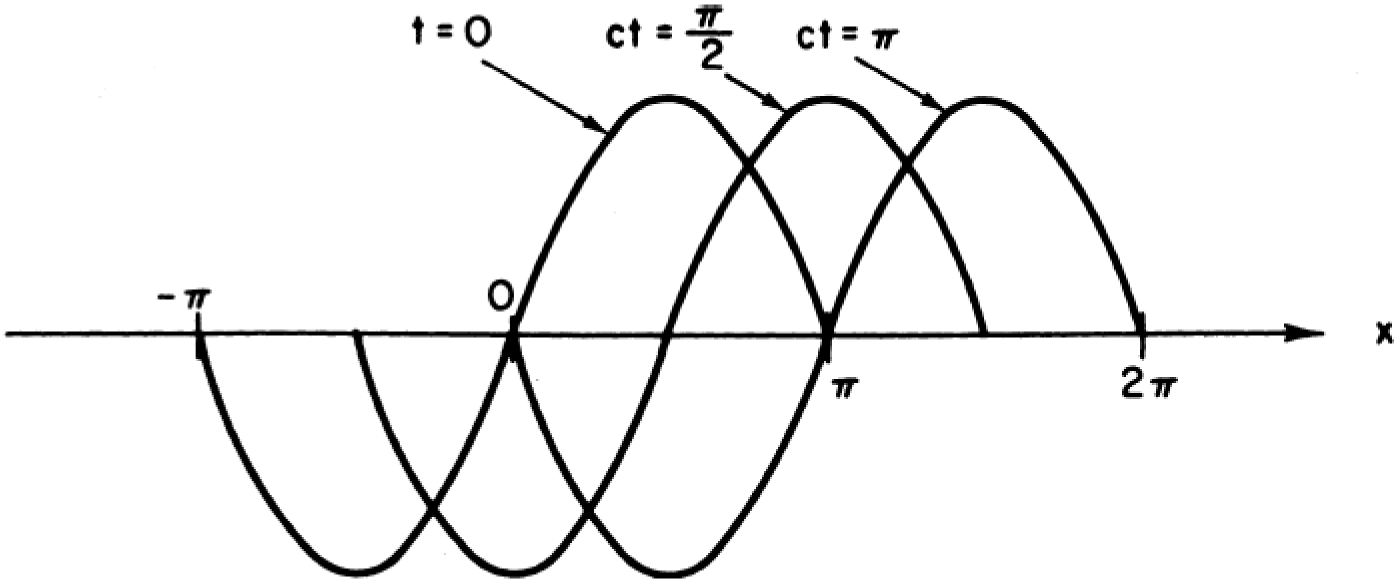
Lecture 5: Waves in Atmosphere



- Perturbation Method
- Properties of Wave
- Shallow Water Gravity Waves
- Rossby Waves

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Perturbation Method

- With this method, all filed variables are separated into two parts: (a) a basic state part and (b) a deviation from the basic state:

$$u(x,t) = \overline{u} + u'(x,t)$$

Basic state
(time and zonal mean)

Perturbation

$$u\frac{\partial u}{\partial x} = \left(\overline{u} + u'\right)\frac{\partial}{\partial x}\left(\overline{u} + u'\right) = \overline{u}\frac{\partial u'}{\partial x} + u'\frac{\partial u'}{\partial x}$$

Basic Assumptions

- ***Assumptions 1***: : the basic state variables must themselves satisfy the governing equations when the perturbations are set to zero.

- ***Assumptions 2:*** the perturbation fields must be small enough so that all terms in the governing equations that involve products of the perturbations can be neglected.

$$|u'/\overline{u}| \ll 1$$

neglected

$$u\frac{\partial u}{\partial x} = \left(\overline{u} + u'\right)\frac{\partial}{\partial x}\left(\overline{u} + u'\right) = \overline{u}\frac{\partial u'}{\partial x} + \cancel{u'\frac{\partial u'}{\partial x}}$$

Example

Original equations

$$\frac{du}{dt} + \alpha \frac{\partial p}{\partial x} = 0$$

$$\frac{dw}{dt} + \alpha \frac{\partial p}{\partial z} + g = 0$$

$$\alpha \frac{dp}{dt} + p\gamma \frac{d\alpha}{dt} = 0$$

$$\alpha \nabla \cdot \mathbf{V} - \frac{d\alpha}{dt} = 0$$

Linearized equations

$$\frac{\partial u'}{\partial t} + U \frac{\partial u'}{\partial x} + \bar{\alpha} \frac{\partial p'}{\partial x} = 0$$

$$\delta_1 \left(\frac{\partial w'}{\partial t} + U \frac{\partial w'}{\partial x} \right) + \bar{\alpha} \frac{\partial p'}{\partial z} - \frac{g\alpha'}{\bar{\alpha}} = 0$$

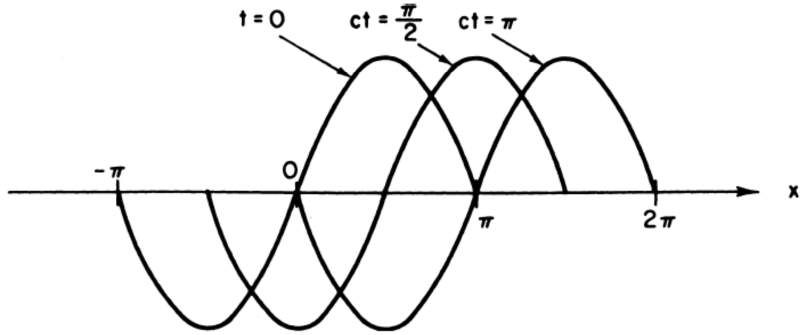
$$\bar{\alpha} \left(\frac{\partial p'}{\partial t} + U \frac{\partial p'}{\partial x} \right) - gw' + \bar{p}\gamma \left(\frac{\partial \alpha'}{\partial t} + U \frac{\partial \alpha'}{\partial x} + w' \frac{\partial \bar{\alpha}}{\partial z} \right) = 0$$

$$\left(\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} \right) \bar{\alpha} - \delta_2 \left(\frac{\partial \alpha'}{\partial t} + U \frac{\partial \alpha'}{\partial x} \right) - w' \frac{\partial \bar{\alpha}}{\partial z} = 0$$

Purpose of Perturbation Method

- If terms that are products of the perturbation variables are neglected, the nonlinear governing equations are reduced to linear differential equations in the perturbation variables in which the basic state variables are specified coefficients.
- These equations can then be solved by standard methods to determine the character and structure of the perturbations in terms of the known basic state.
- For equations with constant coefficients the solutions are sinusoidal or exponential in character.
- Solution of perturbation equations then determines such characteristics as the propagation speed, vertical structure, and conditions for growth or decay of the waves.
- The perturbation technique is especially useful in studying the stability of a given basic state flow with respect to small superposed perturbations.

Wave Motions



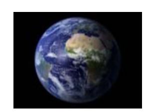
- Perturbations in the atmosphere can be represented in terms of a *Fourier series of* sinusoidal components:

$$\boxed{f(x) = \sum_{s=1}^{\infty} (A_s \sin k_s x + B_s \cos k_s x)}$$

$$k_s = 2\pi s / L$$ ➔ ***zonal wave number (units m^{-1})***

L: distance around a latitude circle,

s: *planetarywave number, an integer designating* the number of waves around a latitude circle



Another Way to Represent Waves

Since $\exp(i\phi) = \cos\phi + i\sin\phi$

Any wave motion can be represented as:

$$f_s(x) = \mathrm{Re}[C_s \exp(ik_s x)]$$
$$= \mathrm{Re}[C_s \cos k_s x + i C_s \sin k_s x]$$

Waves

$$e^{i(kx+ly+nz+vt)}$$

k: zonal wave number

l: meridional wave number

n: vertical wave number

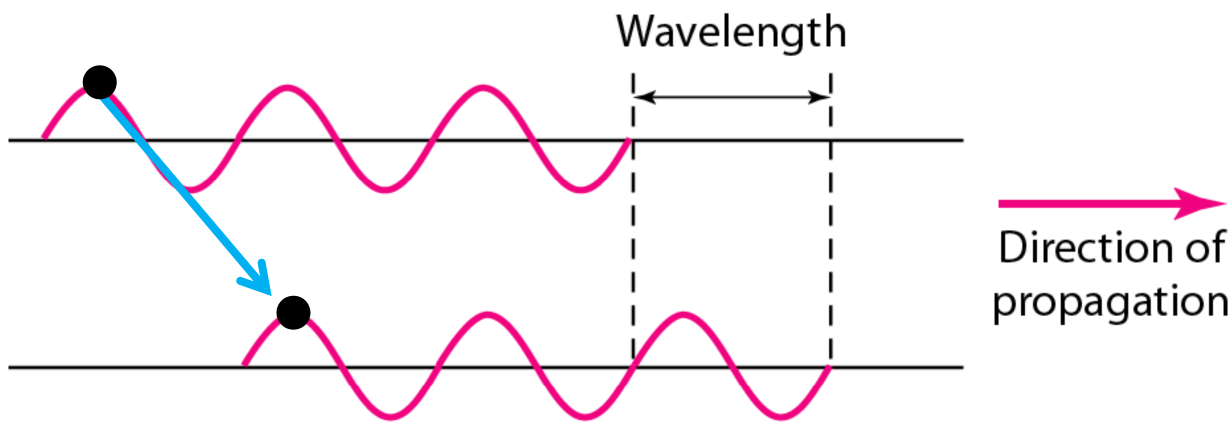
v: frequency

Phase Speed

- The phase velocity of a wave is the rate at which the phase of the wave propagates in space.
- The phase speed is given in terms of the wavelength λ and period T (or frequency ν and wavenumber k) as:

$$c = \frac{\lambda}{T}.$$

$$c = \nu / k$$



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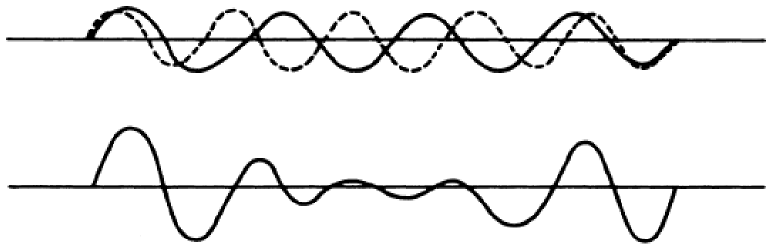
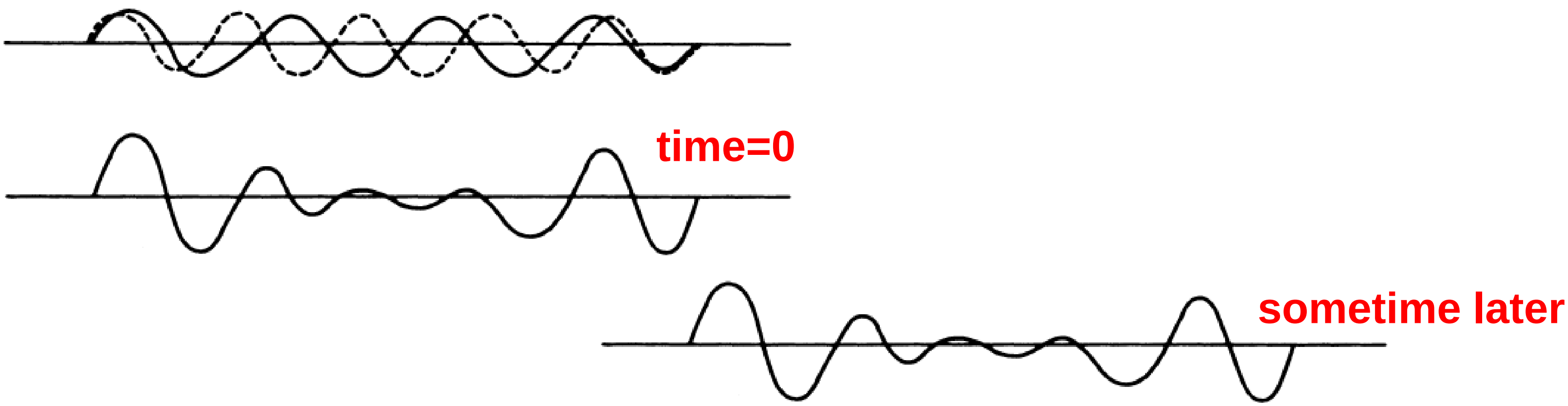
A Group of Waves with Different Wavenumbers



$$\boxed{c = \nu / k}$$

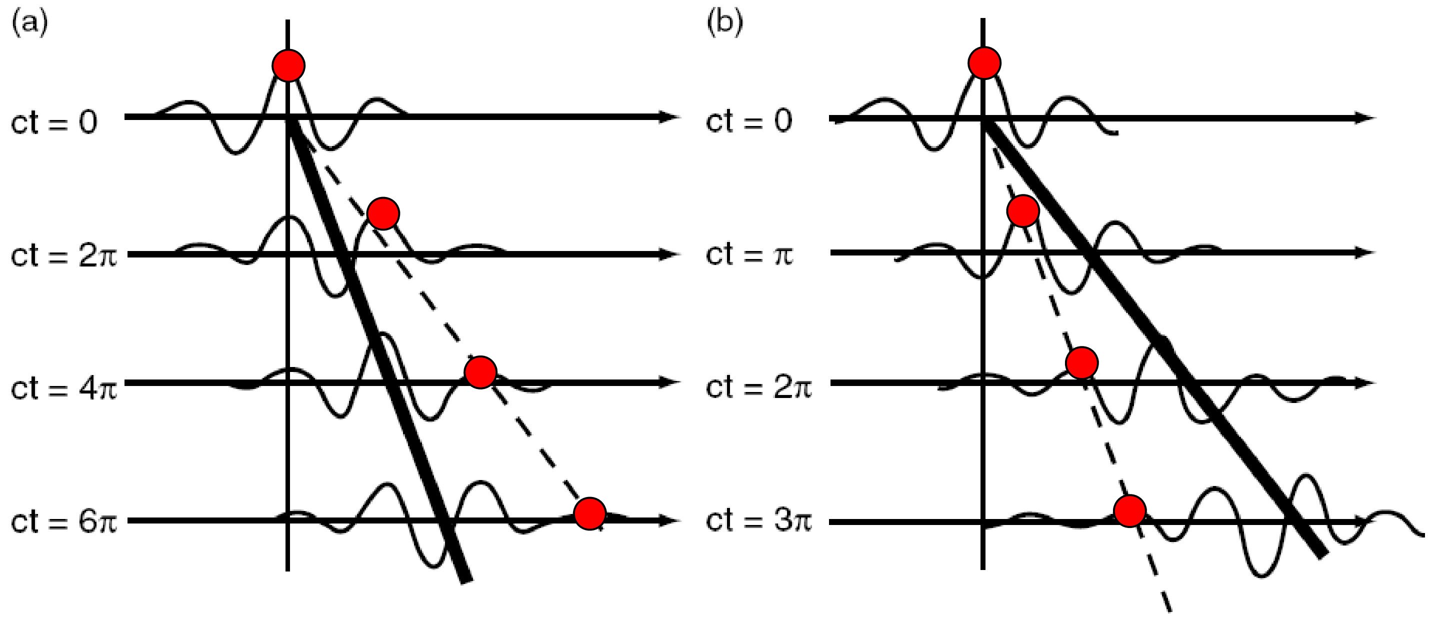
- In cases where several waves add together to form a single wave shape (called the **envelope**), each individual wave component has its own wavenumber and phase speed.
- For waves in which the phase speed varies with k, the various sinusoidal components of a disturbance originating at a given location are at a later time found in different places. Such waves are *dispersive*.
- For *nondispersive* waves, their phase speeds that are independent of the wave number.

Non-Dispersive Waves



- Some types of waves, such as acoustic waves, have phase speeds that are independent of the wave number.
- In such *nondispersive waves a spatially* localized disturbance consisting of a number of Fourier wave components (a *wave group) will preserve its shape as it propagates in space at the phase speed of the* wave.

Dispersive Waves



- For dispersive waves, the shape of a wave group will not remain constant as the group propagates.
- Furthermore, the group generally broadens in the course of time, that is, the energy is *dispersed.*
- When waves are dispersive, the speed of the wave group is generally different from the average phase speed of the individual Fourier components.
- In synoptic-scale atmospheric disturbances, however, the group velocity exceeds the phase velocity.

Group Velocity

$$\boxed{c_{gx} = \partial \nu / \partial k}$$

- The group velocity of a wave is the velocity with which the overall shape of the wave's amplitudes (i.e. envelope) propagates through space.
- Two horizontally propagating waves of equal amplitude but slightly different wavelengths with wave numbers and frequencies differing by $2\delta k$ and $2\delta \nu$, respectively. The total disturbance is thus:

$$\Psi(x,t) = \exp\{i[(k+\delta k)x - (\nu+\delta\nu)t]\} + \exp\{i[(k-\delta k)x - (\nu-\delta\nu)t]\}$$

$$= \left[e^{i(\delta kx - \delta\nu t)} + e^{-i(\delta kx - \delta\nu t)}\right] e^{i(kx-\nu t)}$$

$$= 2\cos(\delta kx - \delta\nu t)\, e^{i(kx-\nu t)}$$

low-frequency amplitude modulation

high-frequency carrier wave

travels at the speed of ***δν/δk*** **➔** ***group velocity***

Restoring Force and Wave

- Waves in fluids result from the action of restoring forces on fluid parcels that have been displaced from their equilibrium positions.
- The restoring forces may be due to compressibility, gravity, ***rotation***, or electromagnetic effects.

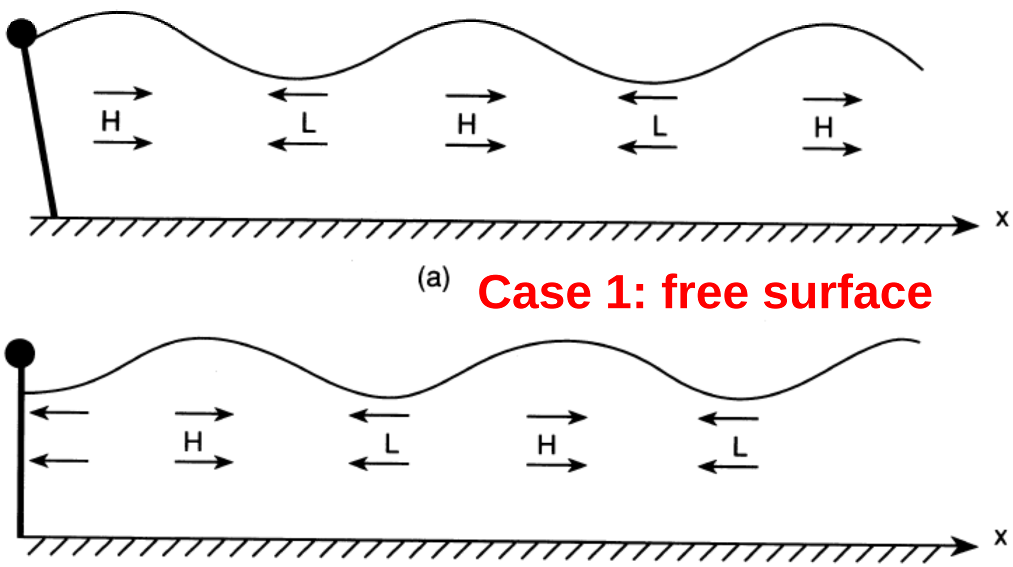
Sound Waves

Rossby Waves

Gravity Waves

Shallow Water Gravity Wave

- Shallow water gravity waves can exist only if the fluid has a free surface or an internal density discontinuity.
- The restoring force is in the vertical so that it is transverse to the direction of propagation.
- The back-and-forth oscillations of the paddle generate alternating upward and downward perturbations in the free surface height, which produce alternating positive and negative accelerations. These, in turn, lead to alternating patterns of fluid convergence and divergence.



Case 1: free surface

diaphragm at the left end. Labels H and L designate centers of high and low perturba
pressure. Arrows show velocity perturbations. (b) The situation 1/4 period later than in (a)

Case 2: internal density discontinuity

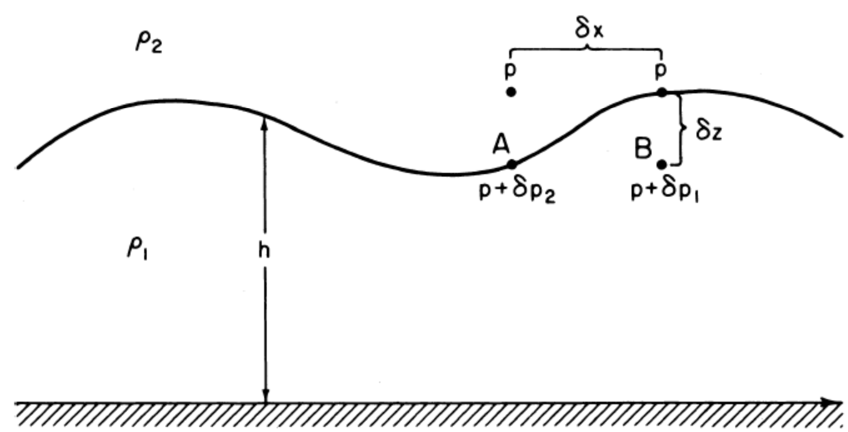


Fig. 7.7 A two-layer fluid system.

Pressure Difference along the Interface

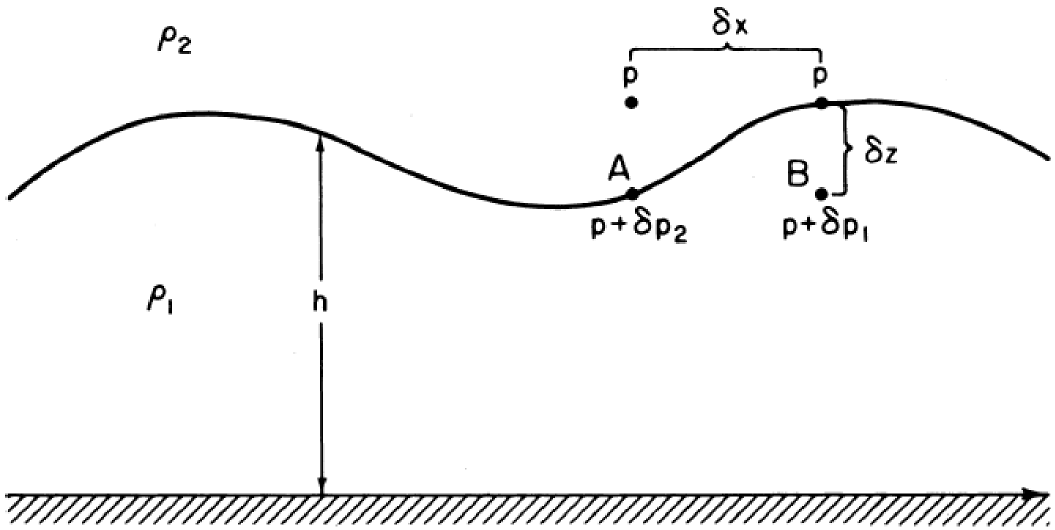


Fig. 7.7 A two-layer fluid system.

$$p + \delta p_1 = p + \rho_1 g \delta z = p + \rho_1 g \left(\partial h / \partial x \right) \delta x$$

$$p + \delta p_2 = p + \rho_2 g \delta z = p + \rho_2 g \left(\partial h / \partial x \right) \delta x$$

$$\lim_{\delta x \to 0} \left[\frac{(p + \delta p_1) - (p + \delta p_2)}{\delta x} \right] = \boxed{g \delta \rho \frac{\partial h}{\partial x}}$$

$$\delta \rho = \rho_1 - \rho_2.$$

Governing Equations for the Shallow Water Case

Momentum Equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{g\delta\rho}{\rho_1}\frac{\partial h}{\partial x}$$

$$u = \overline{u} + u', \quad h = H + h'$$

$$\frac{\partial u'}{\partial t} + \overline{u}\frac{\partial u'}{\partial x} + \frac{g\delta\rho}{\rho_1}\frac{\partial h'}{\partial x} = 0$$

$$\frac{\partial h'}{\partial t} + \overline{u}\frac{\partial h'}{\partial x} + H\frac{\partial u'}{\partial x} = 0$$

Continuity Equation (incompressible)

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$w(h) - w(0) = -h\left(\partial u / \partial x\right)$$

$$w(h) = \frac{Dh}{Dt} = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x}$$

$$w(0) = 0$$

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + h\frac{\partial u}{\partial x} = \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu)$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0$$

Solve for Wave Solutions

$$\frac{\partial u'}{\partial t} + \overline{u}\frac{\partial u'}{\partial x} + \frac{g\delta\rho}{\rho_1}\frac{\partial h'}{\partial x} = 0$$

$$\frac{\partial h'}{\partial t} + \overline{u}\frac{\partial h'}{\partial x} + H\frac{\partial u'}{\partial x} = 0$$

$$\rightarrow$$

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)^2 h' - \frac{gH\delta\rho}{\rho_1}\frac{\partial^2 h'}{\partial x^2} = 0$$

Assume a wave solution:

$$h' = A\exp[ik(x - ct)]$$

We obtain the following relation:

$$c = \overline{u} \pm \left(gH\delta\rho/\rho_1\right)^{1/2}$$

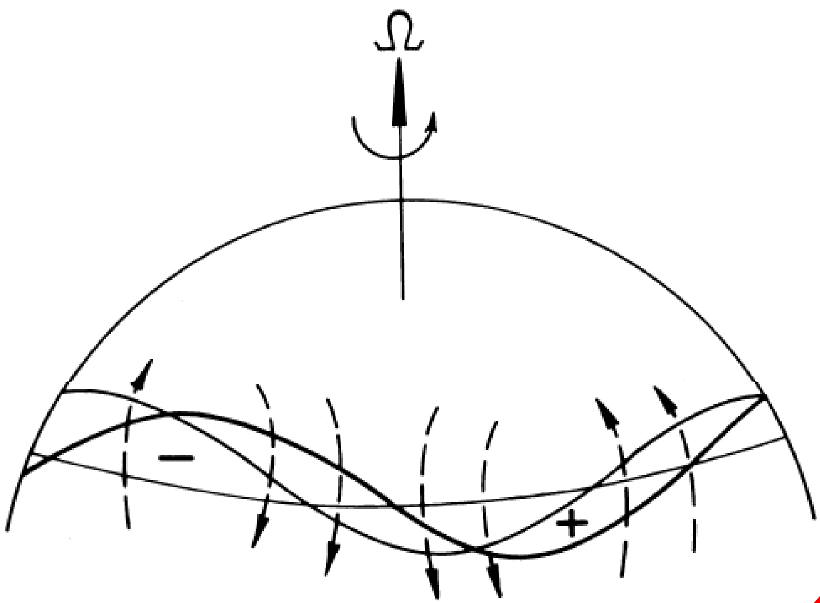
If the upper and lower layers are air and water, respectively, then $\delta\rho \approx \rho_1$

$$\boxed{c = \overline{u} \pm \sqrt{gH}}$$

← Shallow Water Gravity Wave Speed

For an ocean depth of 4 km, the shallow water gravity wave speed is ≈ 200 m/s. (usually caused by earthquake in the ocean).

Rossby Wave



- The wave type that is of most importance for large-scale meteorological processes is the Rossby wave, or planetary wave.

- In an inviscid barotropic fluid of ***constant depth*** (where the divergence of the horizontal velocity must vanish), the Rossby wave is an absolute vorticity-conserving motion that owes its existence to the ***variation of the Coriolis parameter*** with latitude, the so-called β-effect.

$$\boxed{(\zeta + f)/h = \eta/h = \text{Const}}$$

$$\boxed{P \equiv (\zeta_\theta + f)\left(-g\frac{\partial\theta}{\partial p}\right) = \text{Const}}$$

- More generally, in a baroclinic atmosphere, the Rossby wave is a potential vorticity-conserving motion that owes its existence to the ***isentropic gradient of potential vorticity***.

Beta (β) Effect

- Considering a closed chain of fluid parcels initially aligned along a circle of latitude with $\zeta = 0$ at time t_0, then the chain displaced δy from the original latitude at time t_1.

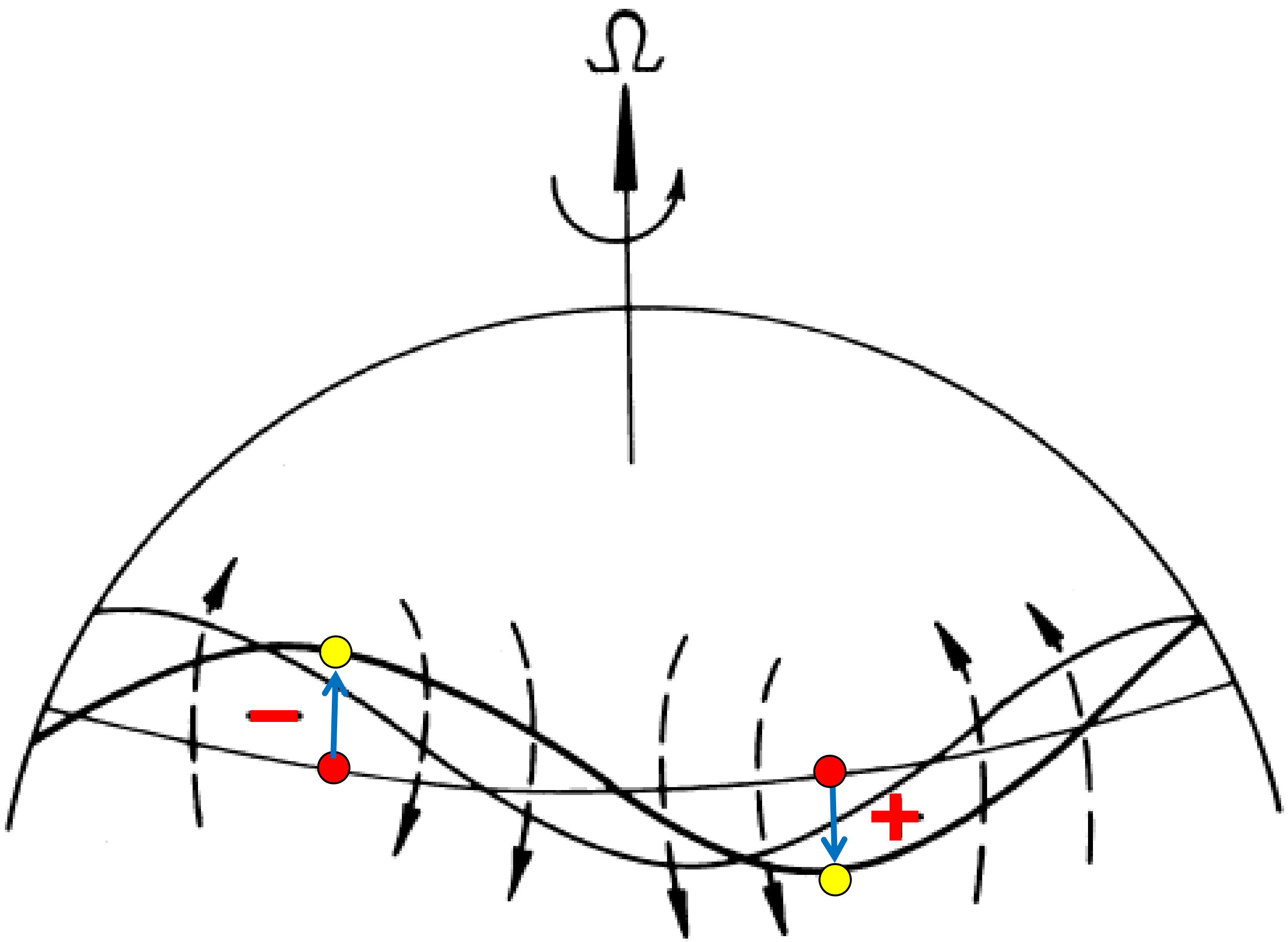
- Due to the conservation of absolute vorticity, we know:

$$(\zeta + f)_{t_1} = f_{t_0}$$

$$\rightarrow \boxed{\zeta_{t_1} = f_{t_0} - f_{t_1} = -\beta \delta y}$$

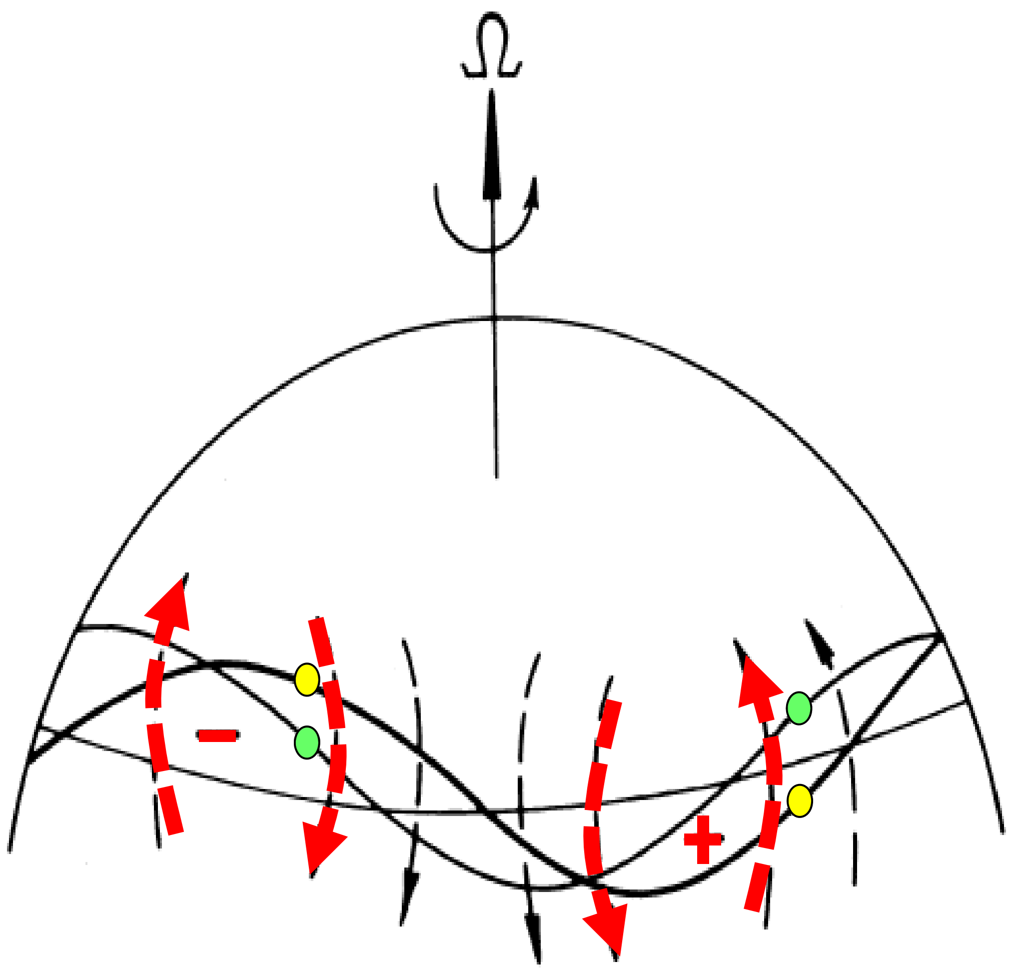
- Here, $\boldsymbol{\beta \equiv df/dy}$ is the planetary vorticity gradient.

β Effect Induces Vorticity



$\zeta_{t_1} = -\beta \delta y < 0$ $\qquad$ $\zeta_{t_1} = -\beta \delta y > 0$

Vorticity Induces Velocity



• This perturbation vorticity field will induce a meridional velocity field, which advects the chain of fluid parcels southward west of the vorticity maximum and northward west of the vorticity minimum.

• Thus, the fluid parcels oscillate back and forth about their equilibrium latitude, and the pattern of vorticity maxima and minima propagates to the west.

Phase Speed of Rossby Wave

$$\boxed{\zeta_{t1} = f_{t0} - f_{t1} = -\beta\delta y}$$

$$\delta y = a \sin\left[k\left(x - ct\right)\right],$$

$$v = D\left(\delta y\right) \big/ Dt = -kca \cos\left[k\left(x - ct\right)\right],$$

$$\zeta = \partial v \big/ \partial x = k^2 ca \sin\left[k\left(x - ct\right)\right]$$

$$\boxed{k^2 ca \sin\left[k\left(x - ct\right)\right] = -\beta a \sin\left[k\left(x - ct\right)\right]}$$

$$\boxed{c = -\beta \big/ k^2}$$

Rossby wave propagates westward

Barotropic Vorticity Equation

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\zeta + \beta v = 0$$

$$u = \overline{u} + u', \quad v = v', \quad \zeta = \partial v'/\partial x - \partial u'/\partial y = \zeta'$$

$$u' = -\partial \psi'/\partial y, \quad v' = \partial \psi'/\partial x \quad \zeta' = \nabla^2 \psi'.$$

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)\nabla^2\psi' + \beta\frac{\partial \psi'}{\partial x} = 0$$

$$\psi' = \mathrm{Re}\left[\Psi \exp(i\phi)\right] \quad \text{where } \phi = kx + ly - \nu t.$$

$$(-\nu + k\overline{u})\left(-k^2 - l^2\right) + k\beta = 0$$

$$\nu = \overline{u}k - \beta k/K^2 \qquad K^2 \equiv k^2 + l^2$$

$$\boxed{c - \overline{u} = -\beta/K^2}$$

Rossby waves are dispersive waves whose phase speeds increase rapidly with increasing wavelength.

Which Direction does Winter Storm Move?

- For a typical midlatitude synoptic-scale disturbance, with similar meridional and zonal scales ($l \approx k$) and zonalwavelength of order 6000 km, the Rossby wave speed relative to the zonal flow is approximately -8 m/s.
- Because the mean zonal wind is generally westerly and greater than 8 m/s, ***synoptic-scale Rossby waves usually move eastward***, but at a phase speed relative to the ground that is somewhat less than the mean zonal wind speed.

Stationary Rossby Wave

- For longer wavelengths the westward Rossby wave phase speed may be large enough to balance the eastward advection by the mean zonal wind so that the resulting disturbance is stationary relative to the surface of the earth.

$$K^2 = \beta / \overline{u} \equiv K_s^2$$