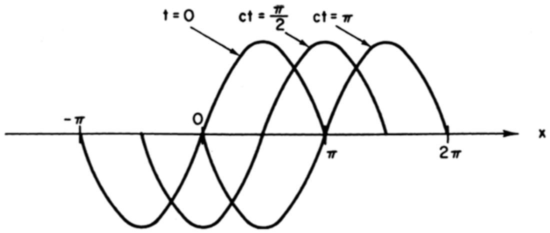
Lecture 5: Waves in Atmosphere



- Perturbation Method
- Properties of Wave
- Shallow Water Model
- Gravity Waves
- Rossby Waves

Waves in the Atmosphere and Oceans

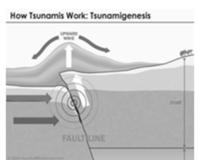
Restoring Force

❑Conservation of potential temperature in the presence of positive static stability
➔ internal gravity waves
❑Conservation of potential vorticity in the presence of a mean gradient of potential vorticity ➔ Rossby waves

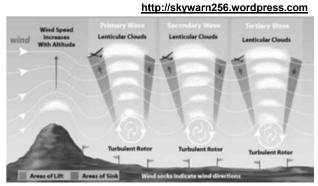
- **External gravity wave** (Shallow-water gravity wave)
- **Internal gravity (buoyancy) wave**
- **Inertial-gravity wave**: Gravity waves that have a large enough wavelength to be affected by the earth's rotation.
- **Rossby Wave**: Wavy motions results from the conservation of potential vorticity.
- **Kelvin wave**: It is a wave in the ocean or atmosphere that balances the Coriolis force against a topographic boundary such as a coastline, or a waveguide such as the equator. Kelvin wave is non-dispersive.

Gravity Waves

http://skywarn256.wordpress.com

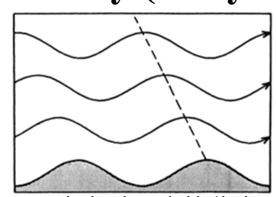


http://www.astronautforhire.com



- **Gravity waves** are waves generated in a fluid medium or at the interface between two media (e.g., the atmosphere and the ocean) which **has the restoring force of gravity or buoyancy**.
- When a fluid element is displaced on an interface or internally to a region with a different density, gravity tries to restore the parcel toward equilibrium resulting in an oscillation about the equilibrium state or wave orbit.
- Gravity waves on an air-sea interface are called surface gravity waves or surface waves while internal gravity waves are called internal waves.

Internal Gravity (Buoyancy) Waves



❑In a fluid, such as the ocean, which is bounded both above and below, gravity waves propagate primarily in the horizontal plane since vertically traveling waves are reflected from the boundaries to form standing waves.

❑ In a fluid that has no upper boundary, such as the atmosphere, gravity waves may propagate vertically as well as horizontally. In vertically propagating waves the phase is a function of height. Such waves are referred to as *internal waves.*

❑**Gravity waves are generated during the adjustment process toward the geostrophic balance/equilibrium**.

❑ *Although internal* gravity waves are not generally of great importance for synoptic-scale weather forecasting (and indeed are nonexistent in the filtered quasi-geostrophic models), they can be important in mesoscale motions.

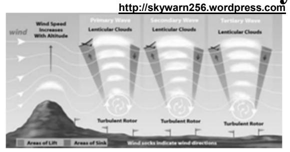
Internal Gravity Waves and Static Stability

In Oceans

$$N = \sqrt{-\frac{g}{\rho_0}\frac{\partial \rho(z)}{\partial z}}$$

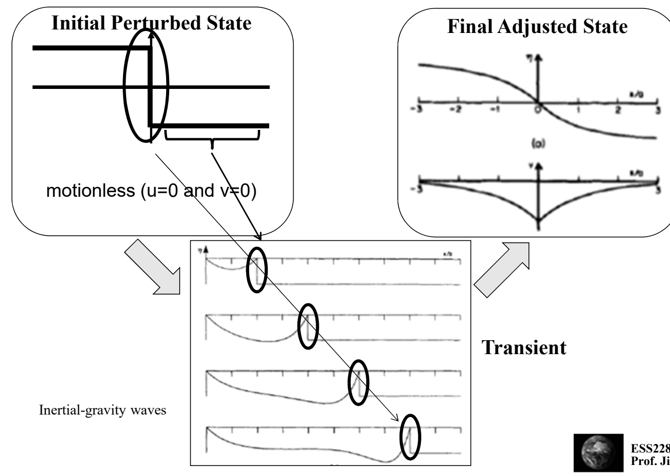
In Atmosphere

$$N \equiv \sqrt{\frac{g}{\theta}\frac{d\theta}{dz}},$$



- **Internal gravity waves can be found in both the *statically stable* (dΘ/dz>0) atmosphere and the *stably stratified* (-dρ/dz>0) ocean.**
- The buoyancy frequency for the internal gravity wave in the ocean is determined by the vertical density gradient, while it is determined by the vertical gradient of potential temperature.
- In the troposphere, the typical value of N is 0.01 sec^{-1}, which correspond to a period of about 10 minutes.
- **Although there are plenty of gravity waves in the atmosphere, most of them have small amplitudes in the troposphere and are not important**, except that the gravity waves generated by flows over mountains. These mountain waves can have large amplitudes.
- **Gravity waves become more important when they propagate into the upper atmosphere** (particularly in the mesosphere) where their amplitudes got amplified due to low air density there.

An Example of Geostrophic Adjustment



Quasi-Geostrophic Approximation

$$\frac{\partial \zeta_g}{\partial t} + \vec{V}_g \bullet \nabla \zeta_g + \beta v_g = -f \nabla \bullet \vec{V}$$

- Quasi-geostrophic approximation use the geostrophic wind for the actual wind everywhere *except* when computing divergence.
- The Q-G approximation eliminates both sound and gravity waves as solutions to the equations of motion.

Why Eliminating Gravity Waves in Numerical Weather/Climate Predictions?

- In mathematics, the Courant–Friedrichs–Lewy (CFL) condition is a necessary condition for convergence while solving certain partial differential equations (usually hyperbolic PDEs) numerically by the method of finite differences.
- It arises in the numerical analysis of explicit time integration schemes, when these are used for the numerical solution. As a consequence, the time step must be less than a certain time in many explicit time-marching computer simulations, otherwise the simulation will produce incorrect results.
- Gravity waves travel too fast and can prevent us from using longer time steps in numerical weather/climate predictions.
- Examples of phase speed for Rossby waves ≈ 20 m/s
  Kelvin waves ≈ 30-60 m/s
  gravity wave (4km ocean) ≈ 200 m/s

Gravity Waves in the Atmosphere

- ❑ In the Earth's atmosphere, gravity waves are a mechanism that produce the transfer of momentum from the troposphere to the stratosphere and mesosphere. ➔ important to the generation of QBO in the stratosphere.
- ❑ **Gravity waves are generated in the troposphere by frontal systems (adiabatic heating) or by airflow over mountains (topographic forcing)**

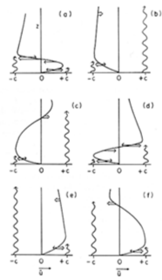


Fig. 8.7. Schematic representation of the evolution of the mean flow in Plumb's analog of the QBO. Six stages of a complete cycle are shown. Double arrows show wave-driven accelerations and single arrows show viscously driven accelerations. Wavy lines indicate relative penetration of easterly and westerly waves. See text for details. [After Plumb (1984).]

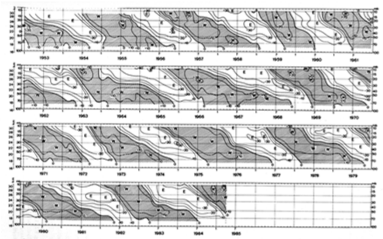


Fig. 8.1. Time-height section of monthly mean zonal winds (m s⁻¹) at equatorial stations (Jan. 1953–Aug. 1967: Canton Island, 3°S/172°W; Sept. 1967–Dec. 1975: Gan/Maldive Islands, 1°S/73°E; Jan. 1976–Apr. 1985: Singapore, 1°N/104°E). Isopleths are at 10-m s⁻¹ intervals. Note the alternating downward propagating westerly (W) and easterly (E) regimes. [From Naujokat (1986), with permission.]

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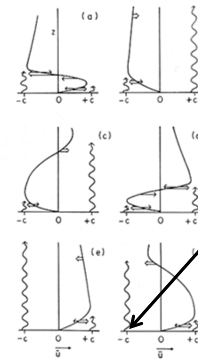
Why QBO?



❑ Kevin Waves accelerate westerly.

❑ Rossby-Gravity Wave accelerate easterly.

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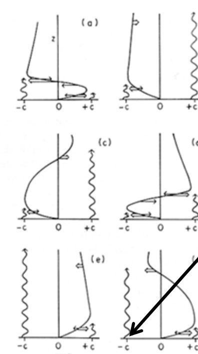
Kelvin Waves

- ❑ The Kelvin wave is a large-scale wave motion of great practical importance in the Earth's atmosphere and ocean.
- ❑ The Kelvin wave is a special type of gravity wave that is affected by the Earth's rotation and trapped at the Equator or along lateral vertical boundaries such as coastlines or mountain ranges.
- ❑ The existence of the Kelvin wave relies on (a) gravity and stable stratification for sustaining a gravitational oscillation, (b) significant Coriolis acceleration, and (c) the presence of vertical boundaries or the equator.
- ❑ There are two basic types of Kelvin waves: boundary trapped and equatorially trapped. Each type of Kelvin wave may be further subdivided into surface and internal Kelvin waves.
- ❑ **Atmospheric** Kelvin waves play **an important role in the adjustment of the tropical atmosphere to convective latent heat release, in the stratospheric quasi-biennial oscillation, and in the generation and maintenance of the Madden–Julian Oscillation**.
- ❑ **Oceanic** Kelvin waves play a critical role in tidal motion, in the adjustment of the tropical ocean to wind stress forcing, and in generating and sustaining the **El Nino Southern Oscillation**.

(from Bin Wang 2002)

Why QBO?



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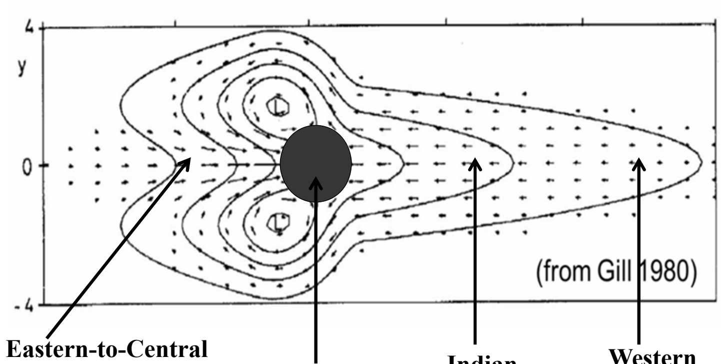
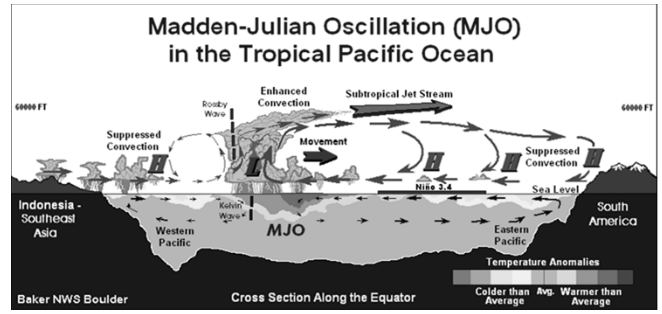
Gill Type Response to Tropical Heating

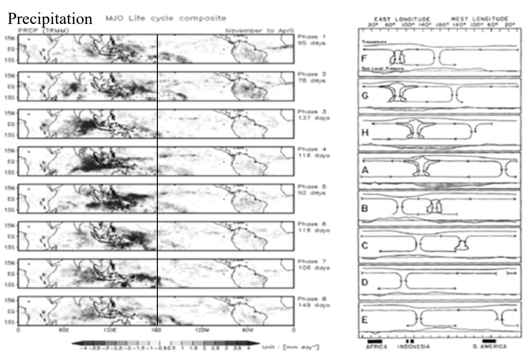


Madden Julian Oscillation (MJO)



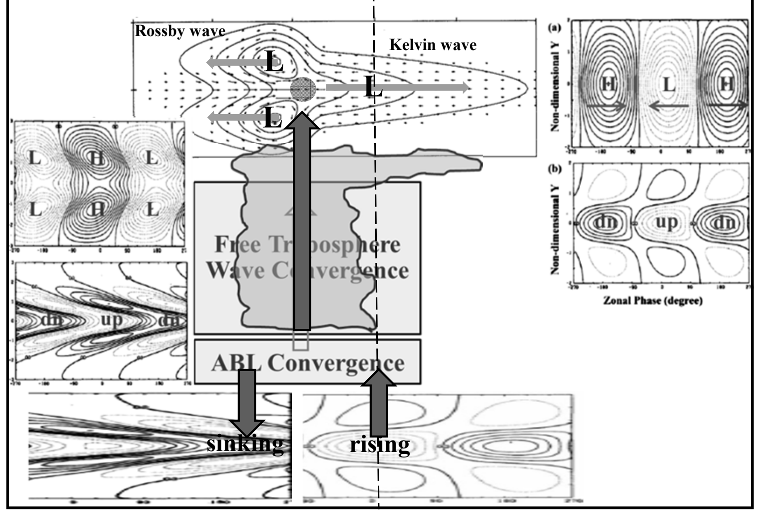
- ❑ Basic Features
- ❑ Generation Mechanisms

MJO: Convection

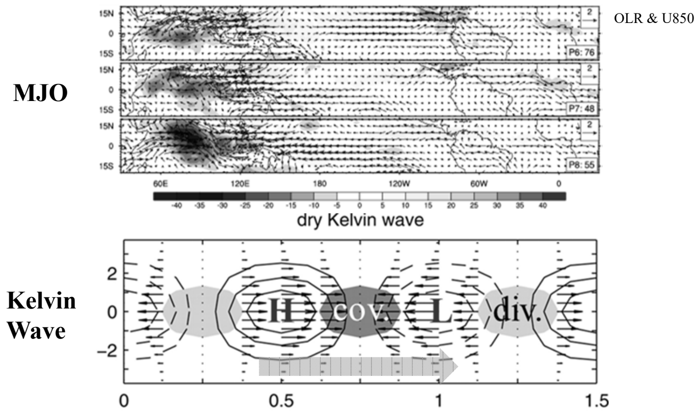


- ❑ The MJO is generally best developed in the region from the southern Indian Ocean eastward across Australia to the western Pacific Ocean
- ❑ The convection and the related cloudiness and precipitation tend to die out east of 180º latitude

Eastward Propagation of the Unstable Mode



Equatorial Kelvin Wave

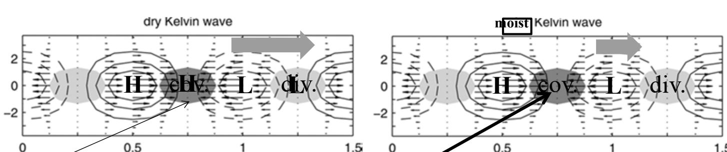


Dry Kelvin wave: C=30-60 m/sec
Moist Kelvin Wave: C=12-15 m/sec
MJO: C=5 m/sec

Using Kelvin Wave to Explain MJO

❑ Need to slow down the propagation speed

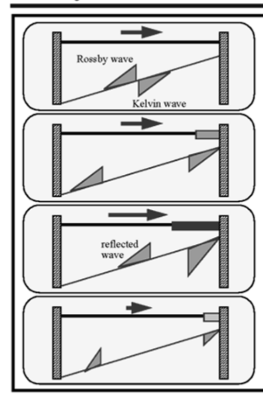
How Moist Convection Slows down Kelvin Waves



dry convergence → accumulation of mass → higher press to the east of H → eastward propagation

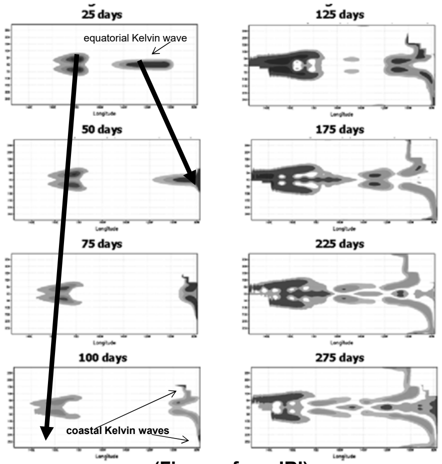
moist convergence → deep convection → cyclonic PV → slow the eastward propagation

Delayed Oscillator Theory



- ❑ Wind forcing at the central Pacific: produces a downwelling Kevin wave propagating eastward and a upwelling Rossby wave propagating westward.
- ❑ wave propagation: the fast kelvin wave causes SST warming at the eastern basin, while slow Rossby wave is reflected at the western boundary.
- ❑ wave reflection: Rossby wave is reflected as a upwelling Kelvin wave and propagates back to the eastern basin to reverse the phase of the ENSO cycle.
- ❑ ENSO period: is determined by the propagation time of the waves.

Wave Propagation and Reflection



(Figures from IRI)

- It takes Kevin wave (phase speed = 2.9 m/s) about 70 days to cross the Pacific basin (17,760km).
- It takes Rossby wave about 200 days (phase speed = 0.93 m/s) to cross the Pacific basin.

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Waves in Shallow, Stably Stratified, Rotational Flows

The synoptic-scale (L=1000 km) and planetary-scale (L=10,000 km) motions we study in the atmosphere (H=10 km) or oceans (H=4km) have the following characteristic:

- **Shallow**: L>>H ➔ Shallow Water Model
- **Stably Stratified**: $d\theta/dz > 0$ ➔ Brunt-Vaisala frequency
- **Rotational:** Rossby number << 1

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Shallow and Deep Water

- "Shallow" in this lecture means that the depth of the fluid layer is small compared with the horizontal scale of the perturbation, i.e., the horizontal scale is large compared with the vertical scale.
- Shallow water gravity waves are the 'long wave approximation" end of gravity waves.
- Deep water gravity waves are the "short wave approximation" end of gravity waves.
- Deep water gravity waves are not important to large-scale motions in the oceans.

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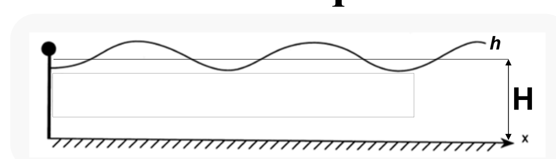
Shallow Water Equation Model



- Consider only a single layer of **incompressible** fluid

  ➔ **conservation of mass**

  ➔ $\boxed{\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0}$
- Shallow ➔ **conservation of momentum**

  ➔ (1) W: vertical velocity is small ➔ hydrostatic balance

  (2) U: **Momentum Equation**

  U independent of z

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \cancel{w\frac{\partial u}{\partial z}} = -\frac{g\delta\rho}{\rho_1}\frac{\partial h}{\partial x}$$

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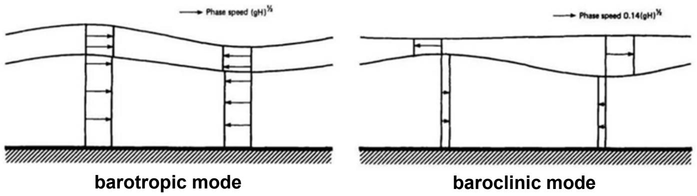
Normal Modes



- The motions corresponding to these particular values of c_c or μ are called ***normal modes*** of oscillation.
- In a system consisting of n layers of different density, there are n normal modes corresponding to the n degrees of freedom.
- A continuously stratified fluid corresponding to an infinite number of layers, and so there is an infinite set of modes.

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Equivalent Depth (H_e)

- An N-layer fluid will have one barotropic mode and (N-1) baroclinic modes of gravity waves, each of which has its own equivalent depth.
- Once the equivalent depth is known, we know the dispersion relation of that mode of gravity wave and we know how fast/slow that gravity wave propagates.

$$\boxed{c_e^2 = gH_e.}$$

- For a continuously stratified fluid, it has an infinite number of modes, but not all the modes are important. We only need to identify the major baroclinic modes and to find out their equivalent depths.

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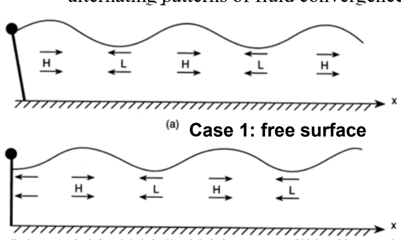
Shallow Water Gravity Wave

- Shallow water gravity waves can exist only if the fluid has a free surface or an internal density discontinuity.
- The restoring force is in the vertical so that it is transverse to the direction of propagation.
- The back-and-forth oscillations of the paddle generate alternating upward and downward perturbations in the free surface height, which produce alternating positive and negative accelerations. These, in turn, lead to alternating patterns of fluid convergence and divergence.



diaphragm at the left end. Labels H and L designate centers of high and low perturba pressure. Arrows show velocity perturbations. (b) The situation 1/4 period later than in (a)

Case 2: internal density discontinuity

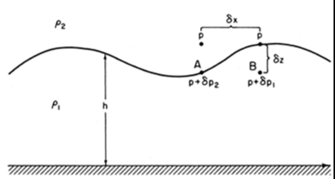


Fig. 7.7 A two-layer fluid system.

Pressure Difference along the Interface

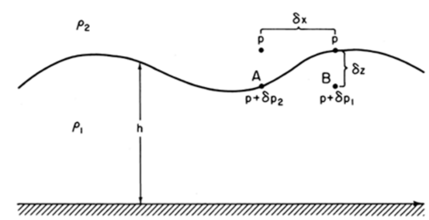


Fig. 7.7 A two-layer fluid system.

$$p + \delta p_1 = p + \rho_1 g \delta z = p + \rho_1 g \left(\partial h / \partial x\right) \delta x$$
$$p + \delta p_2 = p + \rho_2 g \delta z = p + \rho_2 g \left(\partial h / \partial x\right) \delta x$$
$$\lim_{\delta x \to 0}\left[\frac{(p+\delta p_1)-(p+\delta p_2)}{\delta x}\right] = \boxed{g\delta\rho\frac{\partial h}{\partial x}} \qquad \delta\rho = \rho_1 - \rho_2.$$

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Governing Equations for the Shallow Water Case

Momentum Equation

U independent of z

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \cancel{w\frac{\partial u}{\partial z}} = -\frac{g\delta\rho}{\rho_1}\frac{\partial h}{\partial x}$$

$$u = \overline{u} + u', \quad h = H + h'$$

Continuity Equation (incompressible)

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$w(h) - w(0) = -h(\partial u/\partial x)$$

$$w(h) = \frac{Dh}{Dt} = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x}$$

$$w(0) = 0$$

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + h\frac{\partial u}{\partial x} = \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu)$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0$$

$$\boxed{\begin{aligned} &\frac{\partial u'}{\partial t} + \overline{u}\frac{\partial u'}{\partial x} + \frac{g\delta\rho}{\rho_1}\frac{\partial h'}{\partial x} = 0 \\ &\frac{\partial h'}{\partial t} + \overline{u}\frac{\partial h'}{\partial x} + H\frac{\partial u'}{\partial x} = 0 \end{aligned}}$$

Perturbation Method

- With this method, all filed variables are separated into two parts: (a) a basic state part and (b) a deviation from the basic state:

$$u(x,t) = \underbrace{\overline{u}}_{\text{Basic state (time and zonal mean)}} + \underbrace{u'(x,t)}_{\text{Perturbation}}$$

$$u\frac{\partial u}{\partial x} = (\overline{u} + u')\frac{\partial}{\partial x}(\overline{u} + u') = \overline{u}\frac{\partial u'}{\partial x} + u'\frac{\partial u'}{\partial x}$$

Basic Assumptions

- ***Assumptions 1***: : the basic state variables must themselves satisfy the governing equations when the perturbations are set to zero.
- ***Assumptions 2:*** the perturbation fields must be small enough so that all terms in the governing equations that involve products of the perturbations can be neglected.

$$|u'/\overline{u}| \ll 1$$

neglected

$$u\frac{\partial u}{\partial x} = (\overline{u} + u')\frac{\partial}{\partial x}(\overline{u} + u') = \overline{u}\frac{\partial u'}{\partial x} + \cancel{u'\frac{\partial u'}{\partial x}}$$

Example

Original equations

$$\frac{du}{dt} + \alpha\frac{\partial p}{\partial x} = 0$$

$$\frac{dw}{dt} + \alpha\frac{\partial p}{\partial z} + g = 0$$

$$\alpha\frac{dp}{dt} + p\gamma\frac{d\alpha}{dt} = 0$$

$$\alpha\nabla\cdot\mathbf{V} - \frac{d\alpha}{dt} = 0$$

Linearized equations

$$\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + \overline{\alpha}\frac{\partial p'}{\partial x} = 0$$

$$\delta_1\left(\frac{\partial w'}{\partial t} + U\frac{\partial w'}{\partial x}\right) + \overline{\alpha}\frac{\partial p'}{\partial z} - \frac{g\alpha'}{\overline{\alpha}} = 0$$

$$\overline{\alpha}\left(\frac{\partial p'}{\partial t} + U\frac{\partial p'}{\partial x}\right) - gw' + \overline{p}\gamma\left(\frac{\partial \alpha'}{\partial t} + U\frac{\partial \alpha'}{\partial x} + w'\frac{\partial\overline{\alpha}}{\partial z}\right) = 0$$

$$\left(\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z}\right)\overline{\alpha} - \delta_2\left(\frac{\partial \alpha'}{\partial t} + U\frac{\partial \alpha'}{\partial x}\right) - w'\frac{\partial\overline{\alpha}}{\partial z} = 0$$

Purpose of Perturbation Method

- If terms that are products of the perturbation variables are neglected, the nonlinear governing equations are reduced to linear differential equations in the perturbation variables in which the basic state variables are specified coefficients.
- These equations can then be solved by standard methods to determine the character and structure of the perturbations in terms of the known basic state.
- For equations with constant coefficients the solutions are sinusoidal or exponential in character.
- Solution of perturbation equations then determines such characteristics as the propagation speed, vertical structure, and conditions for growth or decay of the waves.
- The perturbation technique is especially useful in studying the stability of a given basic state flow with respect to small superposed perturbations.

Governing Equations for the Shallow Water Case

Momentum Equation

U independent of z

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{g\delta\rho}{\rho_1}\frac{\partial h}{\partial x}$$

$$u = \overline{u} + u', \quad h = H + h'$$

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Solve for Wave Solutions

$$\frac{\partial u'}{\partial t} + \overline{u}\frac{\partial u'}{\partial x} + \frac{g\delta\rho}{\rho_1}\frac{\partial h'}{\partial x} = 0$$

$$\frac{\partial h'}{\partial t} + \overline{u}\frac{\partial h'}{\partial x} + H\frac{\partial u'}{\partial x} = 0$$

$$\longrightarrow \left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)^2 h' - \frac{gH\delta\rho}{\rho_1}\frac{\partial^2 h'}{\partial x^2} = 0$$

Assume a wave solution:

$$h' = A\exp[ik(x - ct)]$$

We obtain the following relation:

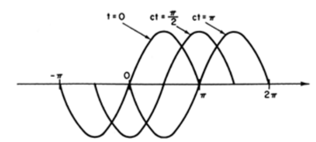
$$c = \overline{u} \pm \left(gH\delta\rho/\rho_1\right)^{1/2}$$

If the upper and lower layers are air and water, respectively, then $\delta\rho \approx \rho_1$

$\boxed{c = \overline{u} \pm \sqrt{gH}}$ ← *Shallow Water Gravity Wave Speed*

For an ocean depth of 4 km, the shallow water gravity wave speed is ≈ 200 m/s. (usually caused by earthquake in the ocean).
200 (meters / second) = 447.387258 mph

Wave Motions



- Perturbations in the atmosphere can be represented in terms of a *Fourier series of* sinusoidal components:

$$f(x) = \sum_{s=1}^{\infty}(A_s \sin k_s x + B_s \cos k_s x)$$

$k_s = 2\pi s/L$ → ***zonal wave number (units m^{-1})***

L: distance around a latitude circle,

s: *planetarywave number, an integer designating* the number of waves around a latitude circle

Another Way to Represent Waves

Since $\exp(i\phi) = \cos\phi + i\sin\phi$

Any wave motion can be represented as:

$$
\begin{aligned}
f_s(x) &= \mathrm{Re}[C_s \exp(ik_s x)] \\
&= \mathrm{Re}[C_s \cos k_s x + i C_s \sin k_s x]
\end{aligned}
$$

Waves

$$e^{\,i(kx+ly+nz+vt)}$$

k: zonal wave number

l: meridional wave number

n: vertical wave number

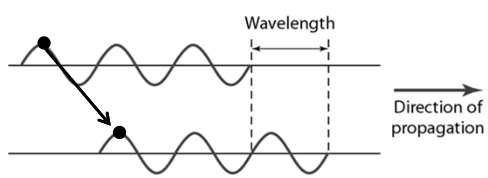
v: frequency

Phase Speed

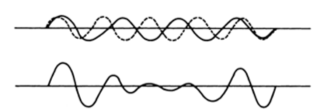
- The phase velocity of a wave is the rate at which the phase of the wave propagates in space.
- The phase speed is given in terms of the wavelength λ and period T (or frequency ν and wavenumber k) as:

$$c = \frac{\lambda}{T}.$$

$$c = \nu/k$$



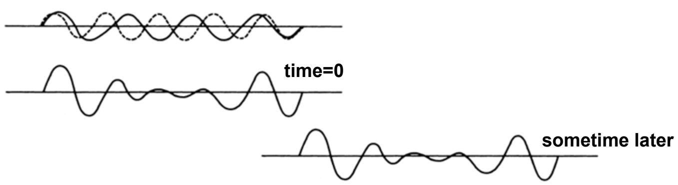
A Group of Waves with Different Wavenumbers



$$\boxed{c = \nu/k}$$

- In cases where several waves add together to form a single wave shape (called the **envelope**), each individual wave component has its own wavenumber and phase speed.
- For waves in which the phase speed varies with k, the various sinusoidal components of a disturbance originating at a given location are at a later time found in different places. Such waves are *dispersive*.
- For *nondispersive* waves, their phase speeds are independent of the wave number.

Non-Dispersive Waves



- Some types of waves, such as acoustic waves, have phase speeds that are independent of the wave number.
- In such *nondispersive waves a spatially* localized disturbance consisting of a number of Fourier wave components (a *wave group) will preserve its shape as it propagates in space at the phase speed of the* wave.

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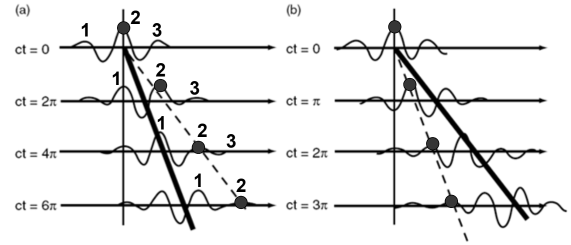
Dispersive Waves



- For dispersive waves, the shape of a wave group will not remain constant as the group propagates.
- Furthermore, the group generally broadens in the course of time, that is, the energy is *dispersed.*
- When waves are dispersive, the speed of the wave group is generally different from the average phase speed of the individual Fourier components.
- In synoptic-scale atmospheric disturbances, however, the group velocity exceeds the phase velocity.

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Group Velocity

$$\boxed{c_{gx} = \partial\nu/\partial k}$$

- The group velocity of a wave is the velocity with which the overall shape of the wave's amplitudes (i.e. envelope) propagates through space.
- Two horizontally propagating waves of equal amplitude but slightly different wavelengths with wave numbers and frequencies differing by $2\delta k$ and $2\delta\nu$, respectively. The total disturbance is thus:

$$\begin{aligned}\Psi(x,t) &= \exp\{i[(k+\delta k)x-(\nu+\delta\nu)t]\}+\exp\{i[(k-\delta k)x-(\nu-\delta\nu)t]\}\\ &= \left[e^{i(\delta kx-\delta\nu t)}+e^{-i(\delta kx-\delta\nu t)}\right]e^{i(kx-\nu t)}\\ &= 2\cos(\delta kx-\delta\nu t)\,e^{i(kx-\nu t)}\end{aligned}$$

low-frequency amplitude modulation

high-frequency carrier wave

travels at the speed of *δν/δk* ➔ *group velocity*

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Solve for Wave Solutions

$$\frac{\partial u'}{\partial t}+\bar{u}\frac{\partial u'}{\partial x}+\frac{g\delta\rho}{\rho_1}\frac{\partial h'}{\partial x}=0$$

$$\frac{\partial h'}{\partial t}+\bar{u}\frac{\partial h'}{\partial x}+H\frac{\partial u'}{\partial x}=0$$

$$\longrightarrow \left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)^2 h'-\frac{gH\delta\rho}{\rho_1}\frac{\partial^2 h'}{\partial x^2}=0$$

Assume a wave solution:

$$h' = A\exp[ik(x-ct)]$$

We obtain the following relation:

$$c=\bar{u}\pm\left(gH\delta\rho/\rho_1\right)^{1/2}$$

If the upper and lower layers are air and water, respectively, then $\delta\rho \approx \rho_1$

$$\boxed{c=\bar{u}\pm\sqrt{gH}}$$

🡐 *Shallow Water Gravity Wave Speed*

For an ocean depth of 4 km, the shallow water gravity wave speed is ≈ 200 m/s. (usually caused by earthquake in the ocean).
200 (meters / second) = 447.387258 mph

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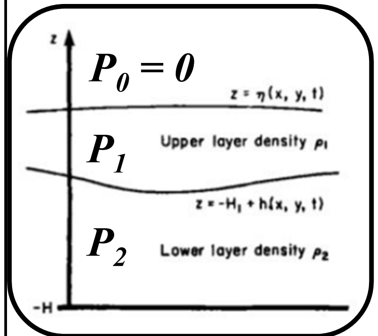
Waves in Shallow, Stably Stratified, Rotational Flows

The synoptic-scale (L=1000 km) and planetary-scale (L=10,000 km) motions we study in the atmosphere (H=10 km) or oceans (H=4km) have the following characteristic:

- ❑ **Shallow**: L>>H ➔ Shallow Water Model
- ❑ **Stably Stratified**: dθ/dz > 0 ➔ Brunt-Vaisala frequency
- ❑ **Rotational:** Rossby number << 1

Two Fluids: *Layer 1* ($-H_1 + h < z < \eta$)



$$p_1 = \rho_1 g(\eta - z)$$

❑ **Momentum Equations**

$$\partial u_1/\partial t = -g\, \partial\eta/\partial x,$$
$$\partial v_1/\partial t = -g\, \partial\eta/\partial y,$$

❑ **Continuity Equation**

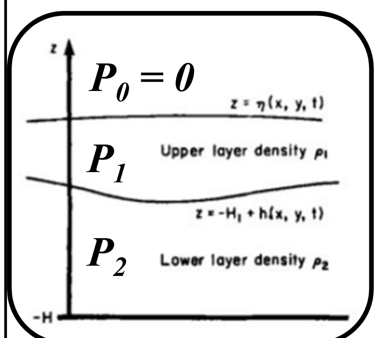
$$\partial(\eta + H_1 - h)/\partial t + H_1(\partial u_1/\partial x + \partial v_1/\partial y) = 0.$$

❑ **Taking time derivative of the continuity equation:**

$$\boxed{\frac{\partial^2}{\partial t^2}(\eta - h) = H_1\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) g\eta \equiv gH_1 \nabla^2\eta}$$

Two Fluids: *Layer 2* ($z < -H_1 + h$)

$$p_2 = \rho_1 g(\eta + H_1 - h) + \rho_2 g(-H_1 + h - z),$$



❑ **Momentum Equations**

$$\frac{\partial u_2}{\partial t} = -\frac{\rho_1}{\rho_2} g \frac{\partial \eta}{\partial x} - g' \frac{\partial h}{\partial x}$$
$$\frac{\partial v_2}{\partial t} = -\frac{\rho_1}{\rho_2} g \frac{\partial \eta}{\partial y} - g' \frac{\partial h}{\partial y}$$

$g' = g(\rho_2 - \rho_1)/\rho_2$
= reduced gravity

❑ **Continuity Equation**

$$\partial h/\partial t + H_2(\partial u_2/\partial x + \partial v_2/\partial y) = 0.$$

❑ **Taking time derivative of the continuity**

$$\boxed{\frac{\partial^2 h}{\partial t^2} = H_2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\rho_1}{\rho_2} g\eta + g'h\right) = H_2 \nabla^2(g\eta - g'\eta + g'h),}$$

Adjustments of the Two-Fluid System

❑ **The adjustments in the two-layer fluid system are governed by:**

$$\frac{\partial^2}{\partial t^2}(\eta - h) = H_1\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) g\eta \equiv gH_1 \nabla^2\eta$$
$$\frac{\partial^2 h}{\partial t^2} = H_2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\rho_1}{\rho_2} g\eta + g'h\right) = H_2 \nabla^2(g\eta - g'\eta + g'h),$$

❑ **Combined these two equations will result in a fourth-order equation, which is difficult to solve.**

❑ **This problem can be greatly simplified by looking for solutions which η and h are proportional:**

$$\boxed{h(x, y, t) = \mu\eta(x, y, t),}$$

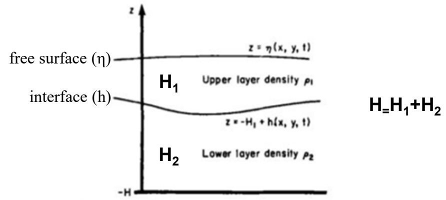
❑ **The governing equations will both reduced to this form:**

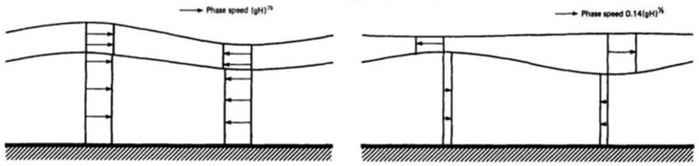
$$\boxed{\partial^2\eta/\partial t^2 = c_e^2 \nabla^2\eta,}$$ provided that

$$\boxed{gH_1/(1 - \mu) = \mu^{-1}(g - g'(1 - \mu))H_2 = c_e^2.}$$

There are two values of μ (and hence two values of c_e) that satisfy this equation.
➔ The motions corresponding to these particular vales are called normal modes of oscillation.

Structures of the Normal Modes





Barotropic mode

$c_0^2 = gH(1 - g'H_1H_2/gH^2\cdots)$,
$\eta/h \approx H/H_2, \quad u_2/u_1 = 1 - g'H_1/gH\cdots$.

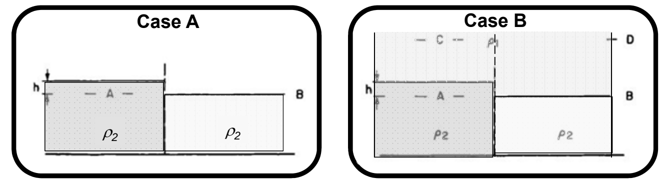
$\eta > h$; u_2&u_1 of same signs

Baroclinic mode

$c_1^2 = (g'H_1H_2/H)(1 + g'H_1H_2/gH^2\cdots)$,
$\eta/h \approx -g'H_2/gH, \quad u_2/u_1 \approx -H_1/H_2$.

$\eta < h$; u_2&u_1 of opposite signs

Reduced Gravity



Pressure difference between A and B:

$\Delta P = \rho_2 * g * h$

Pressure difference between A and B:

$\Delta P = (\rho_2 - \rho_1) * g * h$

The adjustment process in Case B is exactly the same as in the Case A, except the gravitational acceleration is reduced to a value g', where

buoyancy force = density difference * g

$$g' = g(\rho_2 - \rho_1)/\rho_2.$$

Geostrophic Adjustment Problem

shallow water model

$$\frac{\partial u'}{\partial t} - f_0 v' = -g\frac{\partial h'}{\partial x}$$

$$\frac{\partial v'}{\partial t} + f_0 u' = -g\frac{\partial h'}{\partial y}$$

$$\frac{\partial h'}{\partial t} + H\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) = 0$$

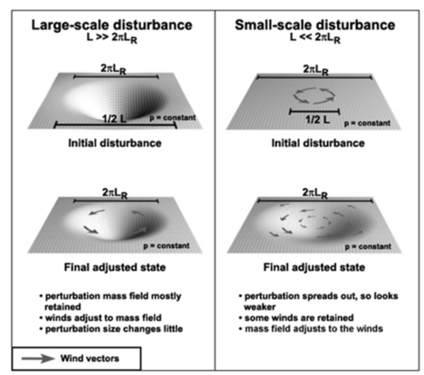
$$\Longrightarrow \frac{\partial^2 h'}{\partial t^2} - c^2\left(\frac{\partial^2 h'}{\partial x^2} + \frac{\partial^2 h'}{\partial y^2}\right) + f_0 H\zeta' = 0$$

$$\frac{\partial \zeta'}{\partial t} + f_0\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) = 0 \Longrightarrow \frac{\partial \zeta'}{\partial t} - \frac{f_0}{H}\frac{\partial h'}{\partial t} = 0 \Longrightarrow \boxed{Q'(x, y, t) = \zeta'/f_0 - h'/H = \text{Const.}}$$

If we know the distribution of perturbation potential vorticity (Q') at the initial time, we know for all time: $Q'(x, y, t) = Q'(x, y, 0)$

And the **final adjusted state** can be determined without solving the time-dependent problem.

Rossby Radius and the Equilibrium State



- If the size of the disturbance is much larger than the Rossby radius of deformation, then the velocity field adjusts to the initial mass (height) field.
- If the size of the disturbance is much smaller than the Rossby radius of deformation, then the mass field adjusts to the initial velocity field.
- If the size of the disturbance is close to the Rossby radius of deformation, then both the velocity and mass fields undergo mutual adjustment.

Restoring Force and Wave

- Waves in fluids result from the action of restoring forces on fluid parcels that have been displaced from their equilibrium positions.
- The restoring forces may be due to compressibility, gravity, ***rotation***, or electromagnetic effects.

Sound Waves

Rossby Waves

Gravity Waves

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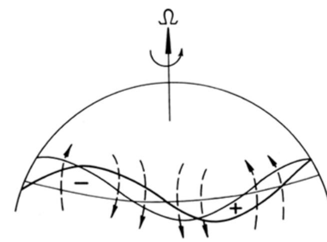
Rossby Wave



- The wave type that is of most importance for large-scale meteorological processes is the Rossby wave, or planetary wave.
- In an inviscid barotropic fluid of ***constant depth*** (where the divergence of the horizontal velocity must vanish), the Rossby wave is an absolute vorticity-conserving motion that owes its existence to the ***variation of the Coriolis parameter*** with latitude, the so-called β-effect.

$$\boxed{(\zeta + f)/h = \eta/h = \text{Const}}$$

$$\boxed{P \equiv (\zeta_\theta + f)\left(-g\frac{\partial\theta}{\partial p}\right) = \text{Const}}$$

- More generally, in a baroclinic atmosphere, the Rossby wave is a potential vorticity-conserving motion that owes its existence to the ***isentropic gradient of potential vorticity***.

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Beta (β) Effect

- Considering a closed chain of fluid parcels initially aligned along a circle of latitude with $\zeta = 0$ at time t_0, then the chain displaced δy from the original latitude at time t_1.
- Due to the conservation of absolute vorticity, we know:

$$(\zeta + f)_{t_1} = f_{t_0}$$

$$\boxed{\zeta_{t_1} = f_{t_0} - f_{t_1} = -\beta\delta y}$$

- Here, $\beta \equiv df/dy$ is the planetary vorticity gradient.

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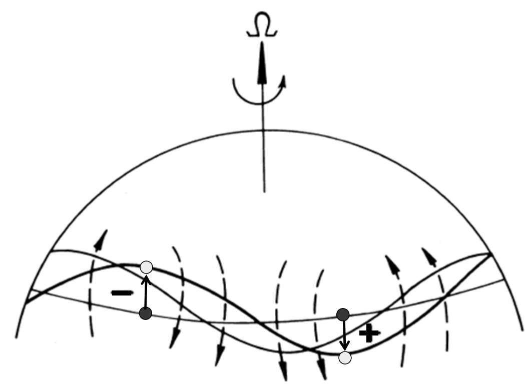
β Effect Induces Vorticity



$\zeta_{t_1} = -\beta\delta y < 0$ $\qquad$ $\zeta_{t_1} = -\beta\delta y > 0$

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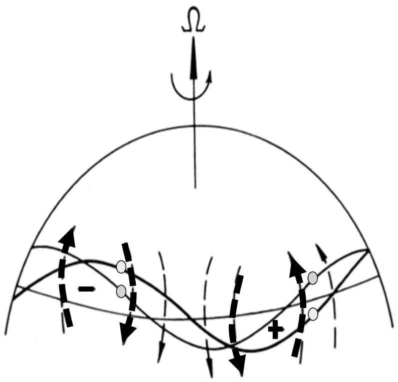
Vorticity Induces Velocity



• This perturbation vorticity field will induce a meridional velocity field, which advects the chain of fluid parcels southward west of the vorticity maximum and northward west of the vorticity minimum.

• Thus, the fluid parcels oscillate back and forth about their equilibrium latitude, and the pattern of vorticity maxima and minima propagates to the west.

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Phase Speed of Rossby Wave

$$\boxed{\zeta_{t1} = f_{t0} - f_{t1} = -\beta\delta y}$$

$$\delta y = a \sin[k(x - ct)],$$

$$v = D(\delta y)/Dt = -kca \cos[k(x - ct)],$$

$$\zeta = \partial v/\partial x = k^2 ca \sin[k(x - ct)]$$

$$\boxed{k^2 ca \sin[k(x - ct)] = -\beta a \sin[k(x - ct)]}$$

$$\boxed{c = -\beta/k^2}$$

Rossby wave propagates westward

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Barotropic Vorticity Equation

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\zeta + \beta v = 0$$

$$u = \overline{u} + u', \quad v = v', \quad \zeta = \partial v'/\partial x - \partial u'/\partial y = \zeta'$$

$$u' = -\partial\psi'/\partial y, \quad v' = \partial\psi'/\partial x \qquad \zeta' = \nabla^2\psi'.$$

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)\nabla^2\psi' + \beta\frac{\partial\psi'}{\partial x} = 0$$

$$\psi' = \mathrm{Re}\left[\Psi\exp(i\phi)\right] \quad \text{where } \phi = kx + ly - \nu t.$$

$$(-\nu + k\overline{u})\left(-k^2 - l^2\right) + k\beta = 0$$

$$\nu = \overline{u}k - \beta k/K^2 \qquad K^2 \equiv k^2 + l^2$$

$$\boxed{c - \overline{u} = -\beta/K^2}$$

Rossby waves are dispersive waves whose phase speeds increase rapidly with increasing wavelength.

Which Direction does Winter Storm Move?

- For a typical midlatitude synoptic-scale disturbance, with similar meridional and zonal scales ($l \approx k$) and zonalwavelength of order 6000 km, the Rossby wave speed relative to the zonal flow is approximately −8 m/s.
- Because the mean zonal wind is generally westerly and greater than 8 m/s, ***synoptic-scale Rossby waves usually move eastward***, but at a phase speed relative to the ground that is somewhat less than the mean zonal wind speed.

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Stationary Rossby Wave

- For longer wavelengths the westward Rossby wave phase speed may be large enough to balance the eastward advection by the mean zonal wind so that the resulting disturbance is stationary relative to the surface of the earth.

$$K^2 = \beta/\overline{u} \equiv K_s^2$$

Dispersive or Non-Dispersive

❑ **Gravity Waves**

Deep-water gravity waves are dispersive

Shallow-water gravity waves are non-dispersive

❑ **Rossby Waves**

dispersive

❑ **Kelvin Waves**

non-dispersive