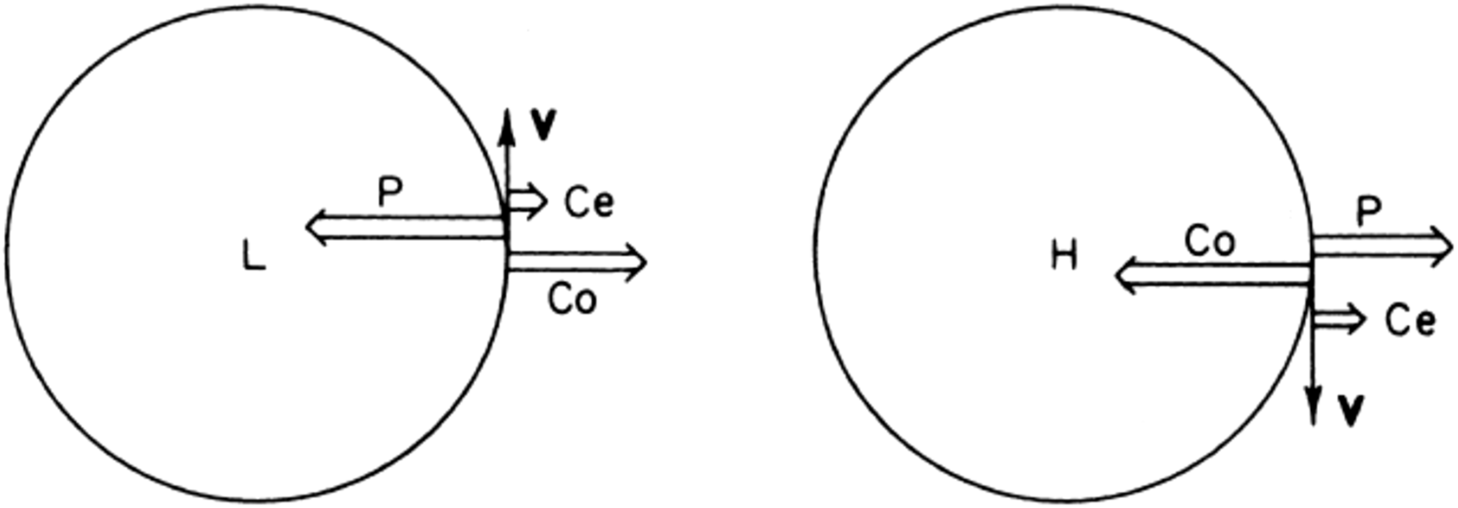
# Lecture 4: Circulation and Vorticity




- Circulation
- Bjerknes Circulation Theorem
- Vorticity
- Potential Vorticity
- Conservation of Potential Vorticity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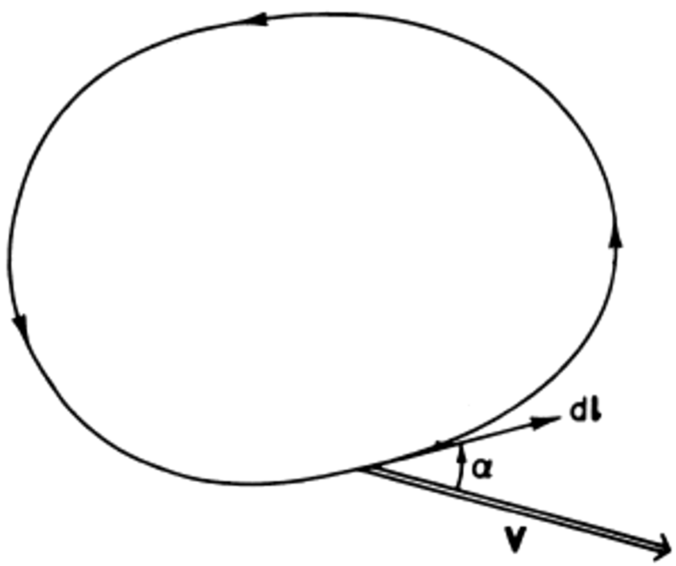
# Measurement of Rotation

- Circulation and vorticity are the two primary measures of rotation in a fluid.
- Circulation, which is a scalar integral quantity, is a *macroscopic* measure of rotation for a finite area of the fluid.
- Vorticity, however, is a vector field that gives a *microscopic* measure of the rotation at any point in the fluid.

# Circulation

- The *circulation*, C, about a closed contour in a fluid is defined as the line integral evaluated along the contour of the component of the velocity vector that is locally tangent to the contour.




$$C \equiv \oint \mathbf{U} \bullet d\mathbf{l} = \oint |\mathbf{U}| \cos \alpha \; dl$$

| | |
|---|---|
| C > 0 ➔ Counterclockwise | |
| C < 0 ➔ Clockwise | |

# Example

- That circulation is a measure of rotation is demonstrated readily by considering a circular ring of fluid of radius R in solid-body rotation at angular velocity $\Omega$ about the z axis.
- In this case, $\mathbf{U} = \Omega \times \mathbf{R}$, where **R** is the distance from the axis of rotation to the ring of fluid. Thus the circulation about the ring is given by:

$$C \equiv \oint \mathbf{U} \bullet d\mathbf{l} = \int_0^{2\pi} \Omega R^2 d\lambda = 2\Omega\pi R^2$$

- In this case the circulation is just $2\pi$ times the angular momentum of the fluid ring about the axis of rotation. Alternatively, note that $C/(\pi R^2) = 2\Omega$ so that the circulation divided by the area enclosed by the loop is just twice the angular speed of rotation of the ring.
- Unlike angular momentum or angular velocity, circulation can be computed without reference to an axis of rotation; it can thus be used to characterize fluid rotation in situations where "angular velocity" is not defined easily.

# Solid Body Rotation

- In fluid mechanics, the state when no part of the fluid has motion relative to any other part of the fluid is called 'solid body rotation'.

# "Meaning" of Circulation

- Circulation can be considered as the amount of force that pushes along a closed boundary or path.
- Circulation is the total "push" you get when going along a path, such as a circle.

# Bjerknes Circulation Theorem

- The circulation theorem is obtained by taking the line integral of Newton's second law for a closed chain of fluid particles.

becomes zero after integration

neglect

$$\oint\left(\frac{D\mathbf{U}}{Dt} = -2\boldsymbol{\Omega}\times\mathbf{U} - \frac{1}{\rho}\nabla p + \mathbf{g} + \mathbf{F}_r\right) dl$$

$$-\nabla\Phi = \mathbf{g} = -g\mathbf{k}.$$

$$-\oint \nabla\Phi \bullet d\mathbf{l}$$

$$\rightarrow \quad \boxed{\frac{DC}{Dt} = -\oint\frac{dp}{\rho} - 2\Omega\frac{DA_e}{Dt}}$$

Term 1 Term 2 Term 3

**Term 1: rate of change of *relative* circulation**

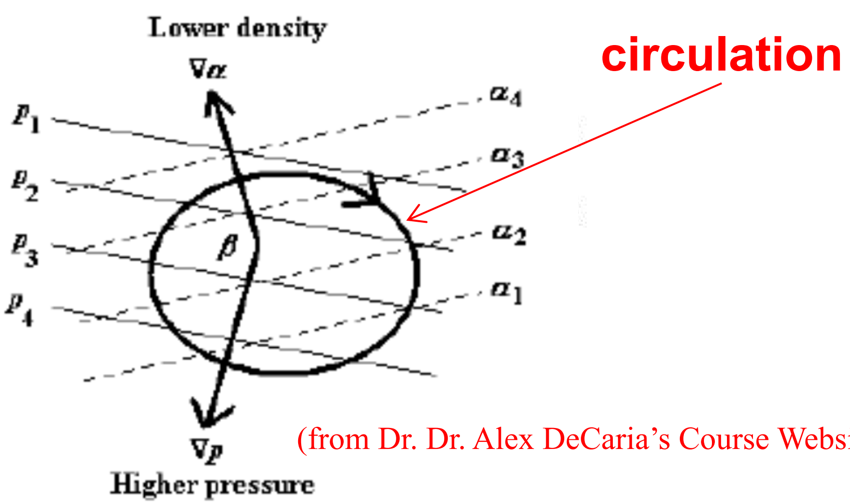
**Term 2: solenoidal term** (for a barotropic fluid, the density is a function only of pressure, and the solenoidal term is zero.)

**Term 3: rate of change of <u>the enclosed area projected on the equatorial plane</u>**

$A_e$

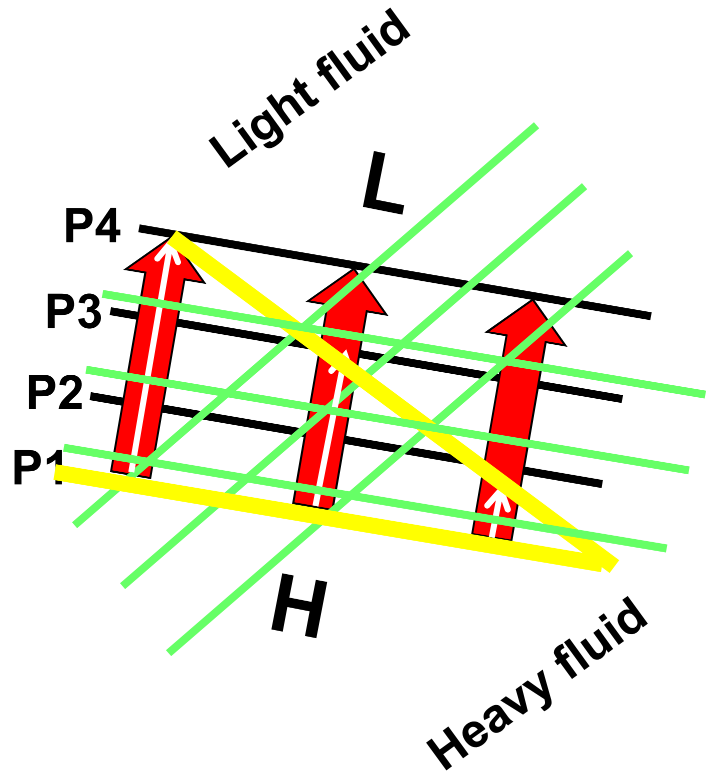
# Solenoidal Term

- In a baroclinic fluid $\oint \alpha\, dp$ is not zero. To evaluate it we remember that
$$dp = \nabla p \bullet d\vec{l}$$
so that we can write
$$\oint \alpha dp = \oint \alpha \nabla p \bullet d\vec{l}$$
which by Stokes Theorem is
$$\oint \alpha \nabla p \bullet d\vec{l} = \int_A \nabla \times (\alpha \nabla p) \bullet d\vec{A}$$
and which expands as
$$\oint \nabla \times (\alpha \nabla p) \bullet d\vec{l} = \int_A \alpha(\nabla \times \nabla p) \bullet d\vec{A} + \int_A (\nabla \alpha \times \nabla p) \bullet d\vec{A}.$$
The first term on the right-hand-side is identically zero. Therefore, we can write Bjerkness' circulation theorem as
$$\frac{DC}{Dt} = -\int_A (\nabla \alpha \times \nabla p) \bullet d\vec{A}. \qquad (2)$$
- The physical meaning of equation (1) is best illustrated if the path of integration in (1) lies in a plane (so that the surface in (2) is flat). In this case
$$-\int_A (\nabla \alpha \times \nabla p) \bullet d\vec{A} = -\int_A |\nabla \alpha| |\nabla p| \sin \beta \, dA$$
where $\beta$ is the angle between the gradients of $\alpha$ and $p$ (see diagram).




(from Dr. Dr. Alex DeCaria's Course Website)

  - In this example $DC/Dt < 0$, so a clockwise circulation would develop as shown below.

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# Solenoidal Term in Baroclinic Flow

- In a baroclinic fluid, circulation may be generated by the pressure-density solenoid term.
- This process can be illustrated effectively by considering the development of a sea breeze circulation,

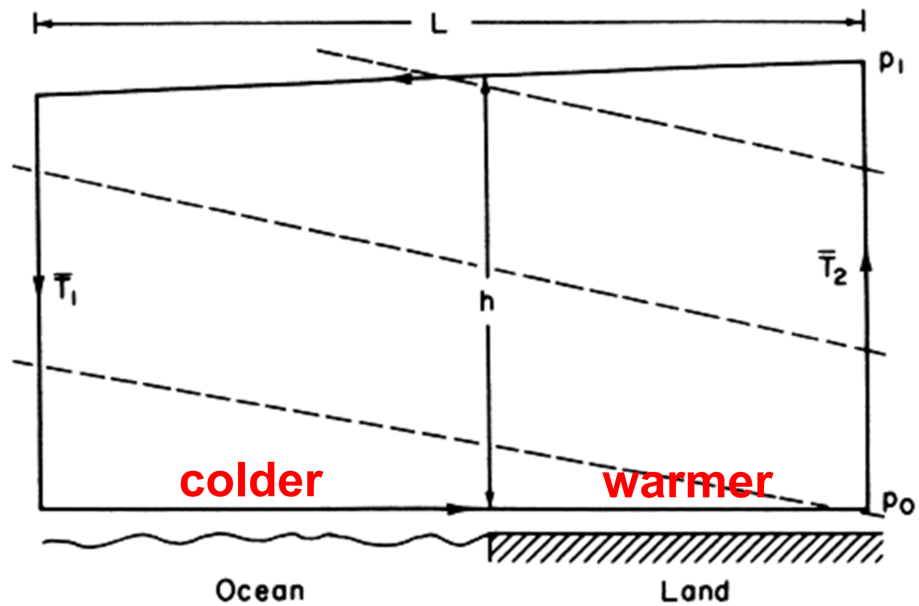
$$\frac{DC_a}{Dt} = -\oint \frac{dp}{\rho}$$

*(P=ρRT)*

$$= -\oint RT\, d\ln p$$

$$= R\ln\left(\frac{p_0}{p_1}\right)(\overline{T}_2 - \overline{T}_1) > 0$$

$$\frac{D\langle v\rangle}{Dt} = \frac{R\ln(p_0/p_1)}{2(h+L)}(\overline{T}_2 - \overline{T}_1)$$




The closed heavy solid line is the loop about which the circulation is to be evaluated. Dashed lines indicate surfaces of constant density.

# What does it mean?

- A counter-clockwise circulation (i.e., sea breeze) will develop in which lighter fluid (the warmer land air; $T_2$) is made to rise and heavier fluid (the colder sea air; $T_1$) is made to sink.
- The effect of this circulation will be to tilt the isopycnals into an oritentation in which they are more nearly parallel with the isobars – that is, toward the barotropic state, in which subsequent circulation change would be zero.
- Such a circulation also lowers the center of mass of the fluid system and thus reduces the potential energy of that system.

# Strength of Sea-Breeze Circulation

$$\frac{D\langle v\rangle}{Dt} = \frac{R\ln(p_0/p_1)}{2(h+L)}(\overline{T}_2 - \overline{T}_1)$$

- Use the following value for the typical sea-land contrast:

  $p_0 = 1000$ hPa

  $p_1 = 900$ hPa

  $T_2 - T_1 = 10^\circ$ C

  $L = 20$ km

  $h = 1$ km

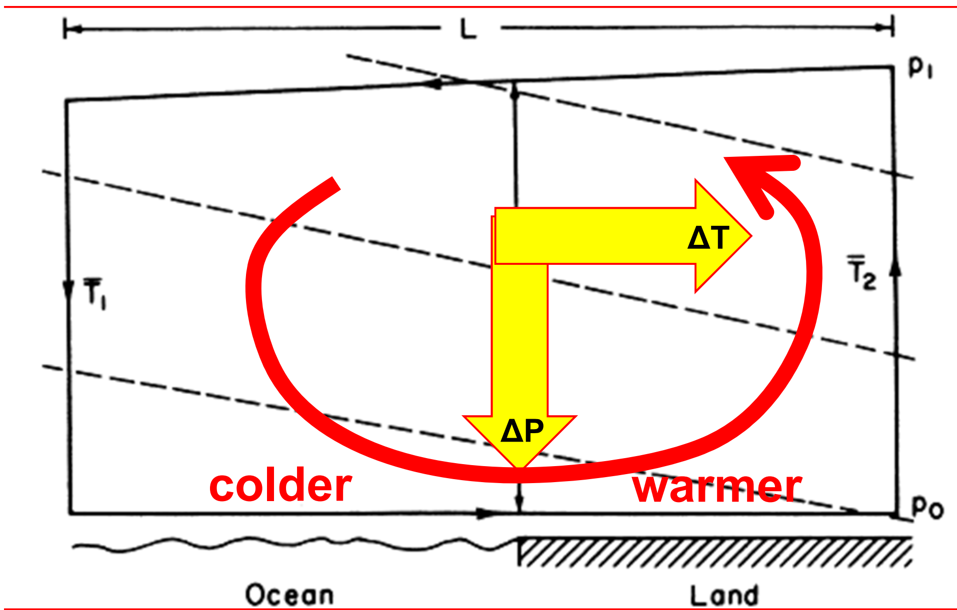
- We obtain an acceleration of about $7 \times 10^{-3}$ $ms^{-2}$ for an acceleration of sea-breeze circulation driven by the solenoidal effect of sea-land temperature contrast.

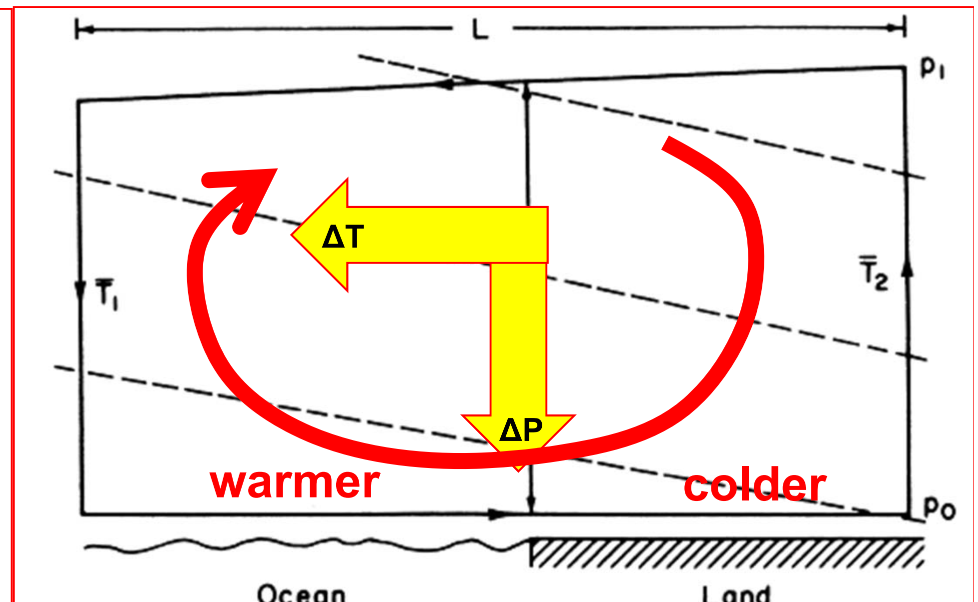
# Solenoidal Term

For an ideal gas, the solenoidal term can be written in terms of the temperature and pressure gradients as

$$\frac{DC}{Dt} = -R' \int_A \nabla T \times \nabla(\ln p) \bullet d\vec{A}.$$

**AM (Sea Breeze)** **PM (Land Breeze)**



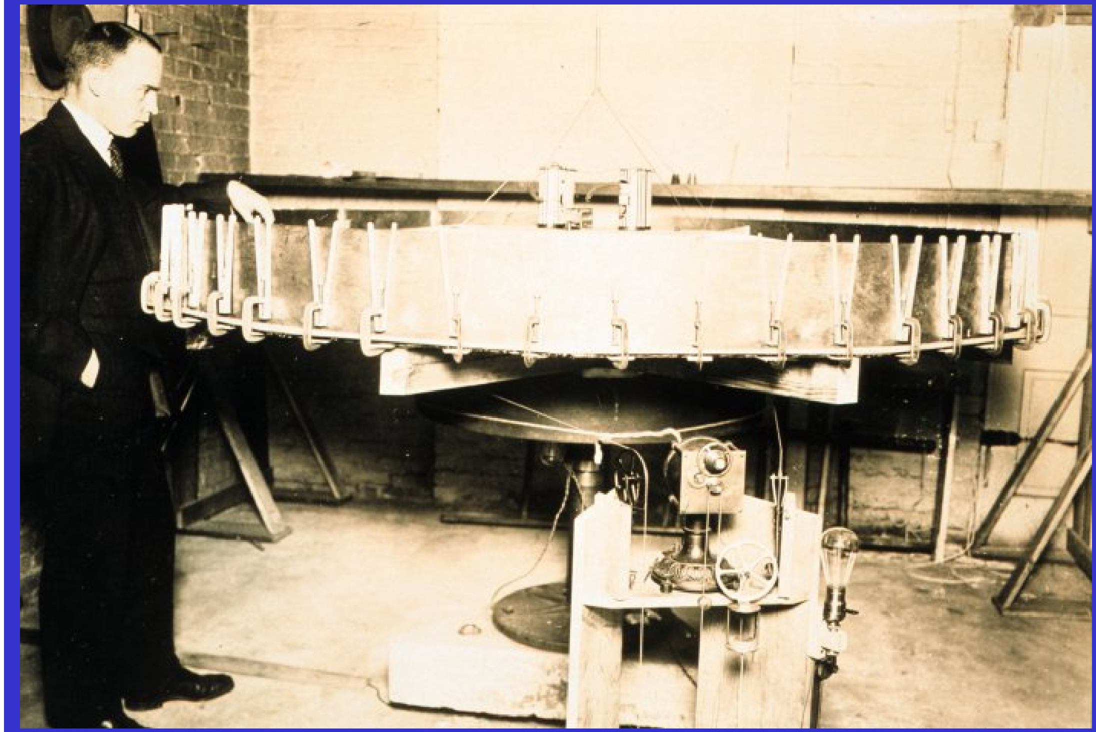



# Polar Front Theory

Vilhelm Bjerknes (1862-1951)

- ***Bjerknes,*** the founder of the Bergen school of meteorology, developed polar front theory during WWI to describe the formation, growth, and dissipation of mid-latitude cyclones.

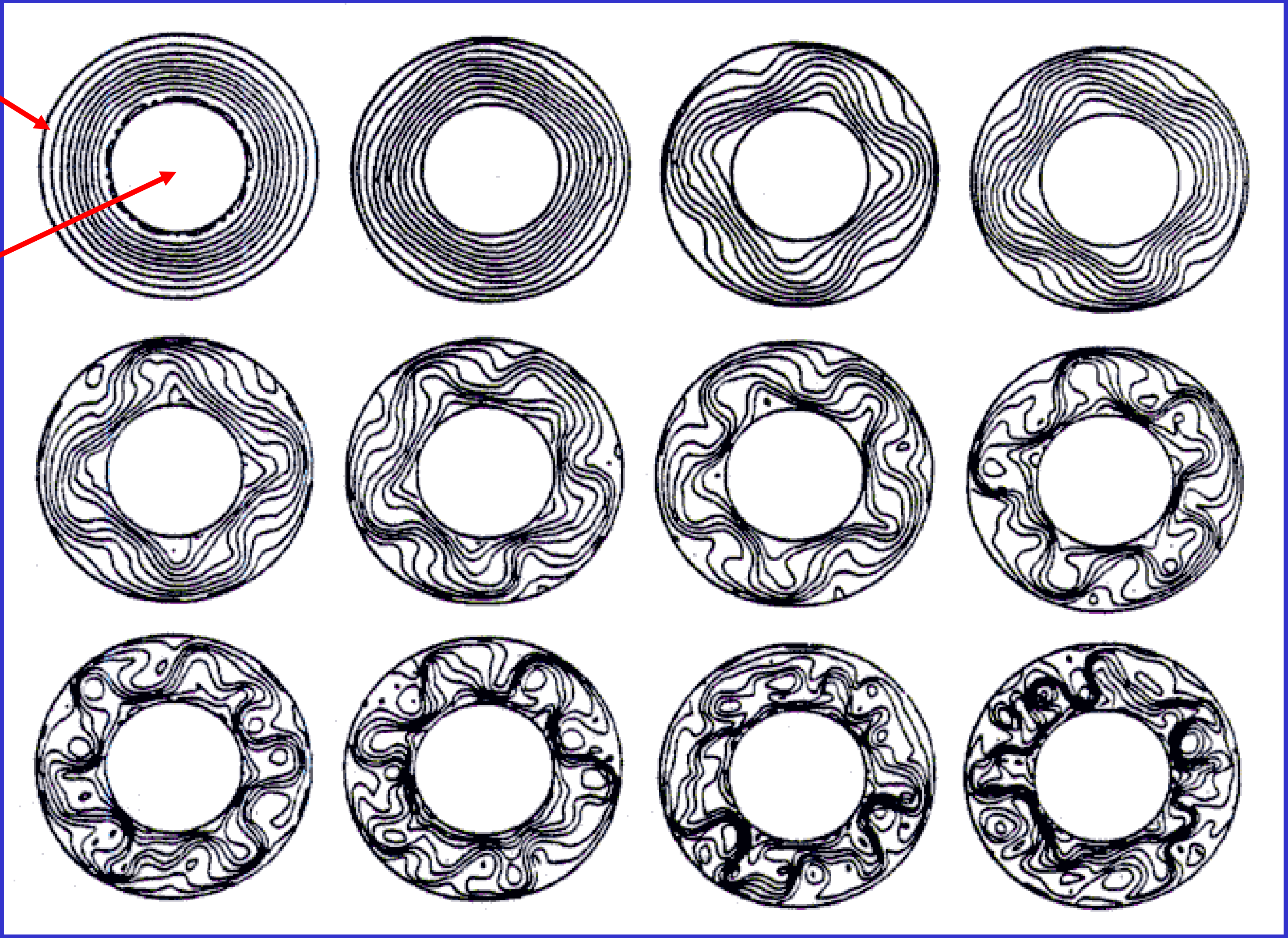
# New Understanding of Cyclone after WWII



Carl Gustav Rossby (1898-1957)

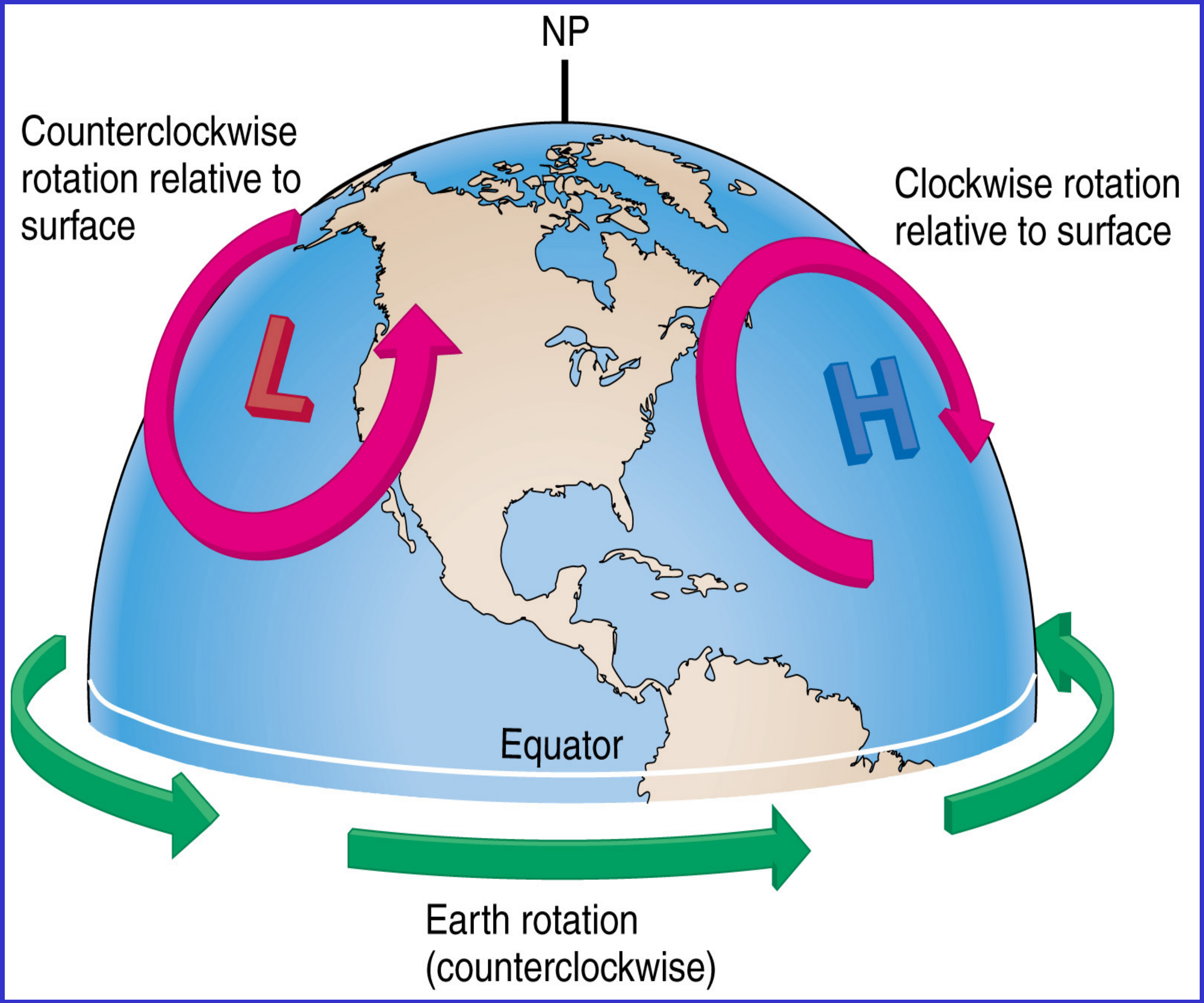
- Carl Rossby mathematically expressed relationships between mid-latitude cyclones and the upper air during WWII.
- Mid-latitude cyclones are a large-scale waves (now called Rossby waves) that grow from the “baroclinic” instabiloity associated with the north-south temperature differences in middle latitudes.

# Rotating Annulus Experiment




(from *"Is The Temperature Rising?"*)

# Rossby Wave

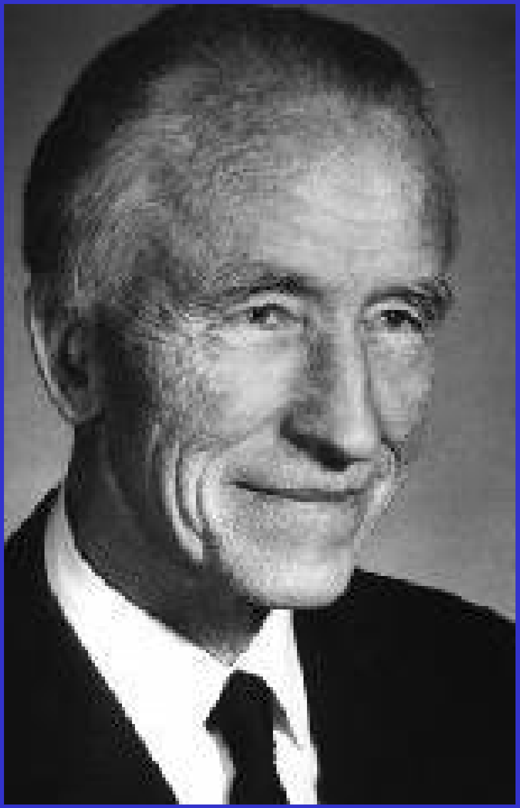

NP
Counterclockwise rotation relative to surface
Clockwise rotation relative to surface
L
H
Equator
Earth rotation (counterclockwise)

# El Nino and Southern Oscillation

- ❑ Jacob Bjerknes was the first one to recognizes that El Nino is not just an oceanic phenomenon (in his 1969 paper).

- ❑ In stead, he hypothesized that the warm waters of El Nino and the pressure seasaw of Walker's Southern Oscillation are part and parcel of the same phenomenon: the ENSO.

- ❑ Bjerknes's hypothesis of coupled atmosphere-ocean instability laid the foundation for ENSO research.



Jacob Bjerknes

# Bjerknes Circulation Theorem

- The circulation theorem is obtained by taking the line integral of Newton's second law for a closed chain of fluid particles.

becomes zero after integration

neglect

$$\oint \left( \frac{D\mathbf{U}}{Dt} = -2\boldsymbol{\Omega} \times \mathbf{U} - \frac{1}{\rho}\nabla p + \mathbf{g} + \mathbf{F}_r \right) dl$$

$$-\nabla\Phi = \mathbf{g} = -g\mathbf{k}.$$

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$$\rightarrow \quad \boxed{\frac{DC}{Dt} = -\oint \frac{dp}{\rho} - 2\Omega \frac{DA_e}{Dt}}$$

Term 1 Term 2 Term 3

**Term 1: rate of change of *relative* circulation**

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**Term 3: rate of change of <u>the enclosed area projected on the equatorial plane</u>**

$A_e$

# Kelvin's Circulation Theorem

- In a barotropic fluid, the solenoid term (Term 2) vanishes.

➔ The absolute circulation ($C_a$) is conserved following the parcel.

$$C_a = C + 2\Omega A_e$$

# Newton's 2nd Law in a Rotating Frame

$$\frac{D_a \mathbf{U}_a}{Dt} = \sum \mathbf{F}$$

using

$$\frac{D_a \mathbf{U}_a}{Dt} = \frac{D\mathbf{U}_a}{Dt} + \mathbf{\Omega} \times \mathbf{U}_a$$

← covert acceleration from an inertial to a rotating frames

$$\mathbf{U}_a = \mathbf{U} + \mathbf{\Omega} \times \mathbf{r}$$

← absolute velocity of an object on the rotating earth is equal to its velocity relative to the earth plus the velocity due to the rotation of the earth

$$\Rightarrow \frac{D_a \mathbf{U}_a}{Dt} = \frac{D}{Dt}(\mathbf{U} + \mathbf{\Omega} \times \mathbf{r}) + \mathbf{\Omega} \times (\mathbf{U} + \mathbf{\Omega} \times \mathbf{r})$$

[Here $\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) = \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{R}) = -\Omega^2 \mathbf{R}$ ]

$$\Rightarrow \frac{D_a \mathbf{U}_a}{Dt} = \frac{D\mathbf{U}}{Dt} + 2\mathbf{\Omega} \times \mathbf{U} - \Omega^2 \mathbf{R}$$

Coriolis force

Centrifugal force

# Kelvin's Circulation Theorem

- In a barotropic fluid, the solenoid term (Term 2) vanishes.

➔ The absolute circulation ($C_a$) is conserved following the parcel.

$$C_a = C + 2\Omega A_e$$

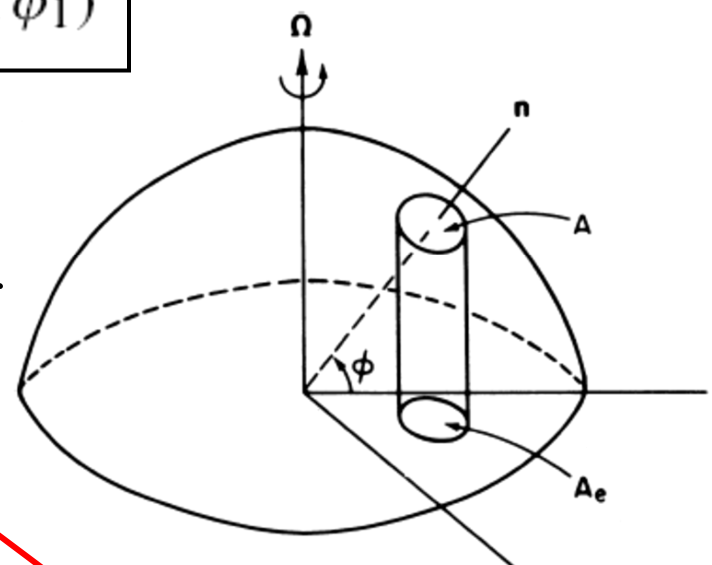
# Applications

*Kelvin's circulation theorem*

- For a barotropic fluid, Bjerknes circulation theorem can be integrated following the motion from an initial state (designated by subscript 1) to a final state (designated by subscript 2), yielding the circulation change:

$$C_2 - C_1 = -2\Omega\,(A_2 \sin\phi_2 - A_1 \sin\phi_1)$$

This equation indicates that in a barotropic fluid the relative circulation for a closed chain of fluid particles will be changed if either the horizontal area enclosed by the loop changes or the latitude changes.




**Coriolis Effect (β effect)**

**divergence effect**

# Example

- Suppose that the air within a circular region of radius 100 km centered at the equator is ***initially motionless*** with respect to the earth. If this circular air mass were moved to the North Pole along an isobaric surface preserving its area, the circulation about the circumference would be:

0

$$C\,(\text{pole}) - C(\text{equator}) = -2\Omega\pi r^2[\sin(\pi/2) - \sin(0)]$$

- Thus the mean tangential velocity at the radius r = 100 km would be:

$$V = C/(2\pi r) = -\,\Omega r \approx -7 \text{ m/sec}$$

- The negative sign here indicates that ***the air has acquired anticyclonic relative circulation***.

# Bjerknes Circulation Theorem

- The circulation theorem is obtained by taking the line integral of Newton's second law for a closed chain of fluid particles.

becomes zero after integration — neglect

$$\oint \left( \frac{D\mathbf{U}}{Dt} = -2\boldsymbol{\Omega} \times \mathbf{U} - \frac{1}{\rho}\nabla p + \mathbf{g} + \mathbf{F}_r \right) dl$$

$$-\nabla\Phi = \mathbf{g} = -g\mathbf{k}, \quad -\oint \nabla\Phi \bullet d\mathbf{l}$$

$$\rightarrow \quad \frac{DC}{Dt} = -\oint \frac{dp}{\rho} - 2\Omega \frac{DA_e}{Dt}$$

Term 1 — Term 2 — Term 3

$$f\nabla_H \bullet \bar{V} + \beta v$$

**Term 1: rate of change of *relative* circulation**

**Term 2: solenoidal term** (for a barotropic fluid, the density is a function only of pressure, and the solenoidal term is zero.)

**Term 3: rate of change of <u>the enclosed area projected on the equatorial plane</u>**

$A_e$

(divergence term + beta effect term)

# Vorticity

- Vorticity is the tendency for elements of the fluid to "spin.“.
- Vorticity can be related to the amount of “circulation” or "rotation" (or more strictly, the local angular rate of rotation) in a fluid.
- Definition:

Absolute Vorticity → $\boldsymbol{\omega}_a \equiv \boldsymbol{\nabla} \times \mathbf{U}_a$

Relative Vorticity → $\boldsymbol{\omega} \equiv \boldsymbol{\nabla} \times \mathbf{U}$

$$\boldsymbol{\omega} = \left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$

# Vertical Component of Vorticity

- In large-scale dynamic meteorology, we are in general concerned only with the vertical components of absolute and relative vorticity, which are designated by ***η*** and ***ζ*** , respectively.

$$\eta \equiv \mathbf{k} \bullet (\nabla \times \mathbf{U}_a), \qquad \zeta \equiv \mathbf{k} \bullet (\nabla \times \mathbf{U})$$

$$\boxed{\eta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f} \qquad \boxed{\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y},}$$

# Vorticity and Circulation

$$\zeta \equiv \lim_{A \to 0} \left( \oint \mathbf{V} \bullet d\mathbf{l} \right) A^{-1}$$

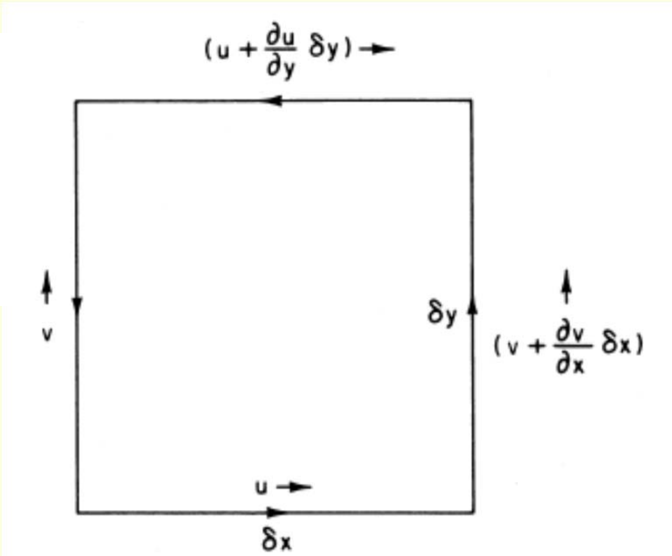
The vertical component of vorticity is defined as the circulation about a closed contour in the horizontal plane divided by the area enclosed, in the limit where the area approaches zero.

$$\delta C = u\delta x + \left( v + \frac{\partial v}{\partial x}\delta x \right) \delta y - \left( u + \frac{\partial u}{\partial y}\delta y \right) \delta x - v\delta y$$

$$= \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \delta x \delta y$$

Dividing through by the area $\delta A = \delta x \delta y$ gives

$$\frac{\delta C}{\delta A} = \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \equiv \zeta$$



# Using Stokes' Theorem

$$\oint \mathbf{U} \bullet d\mathbf{l} = \iint_A (\nabla \times \mathbf{U}) \bullet \mathbf{n}\, dA$$

- Stokes'theorem states that the circulation about any closed loop is equal to the integral of the normal component of vorticity over the area enclosed by the contour.
- For a finite area, circulation divided by area gives the *average* normal component of vorticity in the region.
- Vorticity may thus be regarded as a measure of the local angular velocity of the fluid.

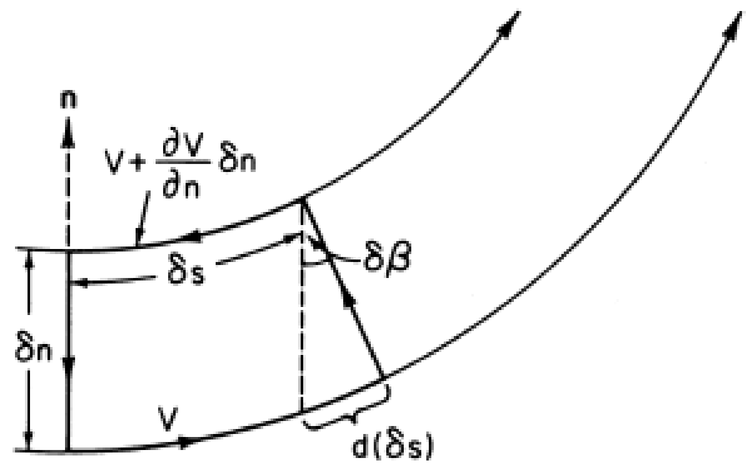
# Vorticity in Natural Coordinate

- Vorticity can be associated with only two broad types of flow configuration.
- It is easier to demonstrate this by considering the vertical component of vorticity in natural coordinates.

$$\delta C = V[\delta s + d(\delta s)] - \left(V + \frac{\partial V}{\partial n}\delta n\right)\delta s$$

$$d(\delta s) = \delta\beta\,\delta n$$

$$\delta C = \left(-\frac{\partial V}{\partial n} + V\frac{\delta\beta}{\delta s}\right)\delta n\,\delta s \qquad \frac{\partial\beta}{\partial s} = \frac{1}{R_s}$$

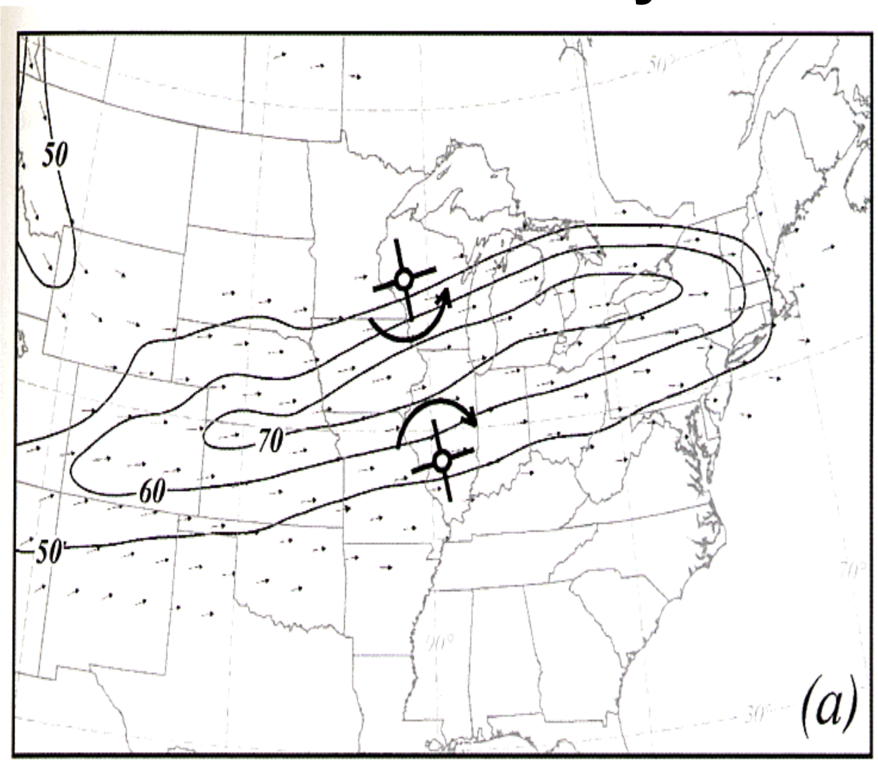



$$\zeta = \lim_{\delta n, \delta s \to 0} \frac{\delta C}{(\delta n\,\delta s)} = -\frac{\partial V}{\partial n} + \frac{V}{R_s}$$

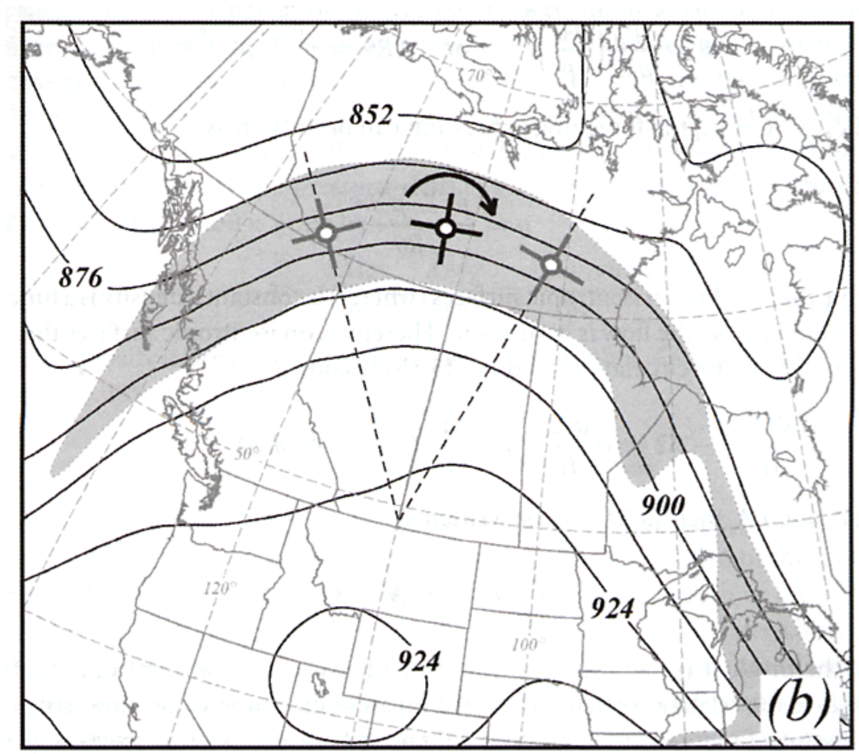
**shear vorticity** **curvature vorticity**

# Vorticity-Related Flow Patterns

## Shear Vorticity




## Curvature Vorticity




Even straight-line motion may have vorticity if the speed changes normal to the flow axis.

**(a) 300mb isotachs; (b) 300mb geopotential hights**

# Potential Vorticity

- We begin with the "circulation equation" (Bjerknes circulation theorem)

$$\frac{DC}{Dt} = -\oint \frac{dp}{\rho} - 2\Omega \frac{DA_e}{Dt}$$

( where $Ae = A\, sin\Phi$)

- We then make use of definitions of potential temperature (Θ) and vorticity (ζ)

$$\theta = T\,(p_s/p)^{R/c_p} \rightarrow \rho = p^{c_v/c_p}(R\theta)^{-1}(p_s)^{R/c_p}$$

$$\rightarrow \oint \frac{dp}{\rho} \propto \oint dp^{(1-c_v/c_p)} = 0$$

$$C \approx \zeta\,\delta A$$

- Therefore, on isentropic surface, there is no solenoid term.

$$\delta A(\zeta_\theta + f) = \text{Const}$$

( where $f = 2\Omega \sin\phi$ )

$$\delta A = -\frac{\delta Mg}{\delta p} = \left(-\frac{\delta\theta}{\delta p}\right)\left(\frac{\delta Mg}{\delta\theta}\right) = \text{Const} \times g\left(-\frac{\delta\theta}{\delta p}\right)$$

$$P \equiv (\zeta_\theta + f)\left(-g\frac{\partial\theta}{\partial p}\right) = \text{Const}$$

**for adiabatic processes**

# *Ertel's Potential Vorticity*

$$P \equiv (\zeta_\theta + f)\left(-g\frac{\partial \theta}{\partial p}\right)$$

- The quantity P [units: $K\ kg^{-1}\ m^2\ s^{-1}$] is the isentropic coordinate form of *Ertel's potential vorticity*.
- *It is defined with a minus sign so that its value is normally* positive in the Northern Hemisphere.
- Potential vorticity is often expressed in the potential vorticity unit (PVU), where 1 PVU = $10^{-6}\ K\ kg^{-1}\ m^2\ s^{-1}$.
- Potential vorticity is always in some sense a measure of the ratio of the absolute vorticity to the ***effective depth of the vortex***.
- The effective depth is just the differential distance between potential temperature surfaces measured in pressure units $(-\partial\theta/\partial p)^{-1}$.

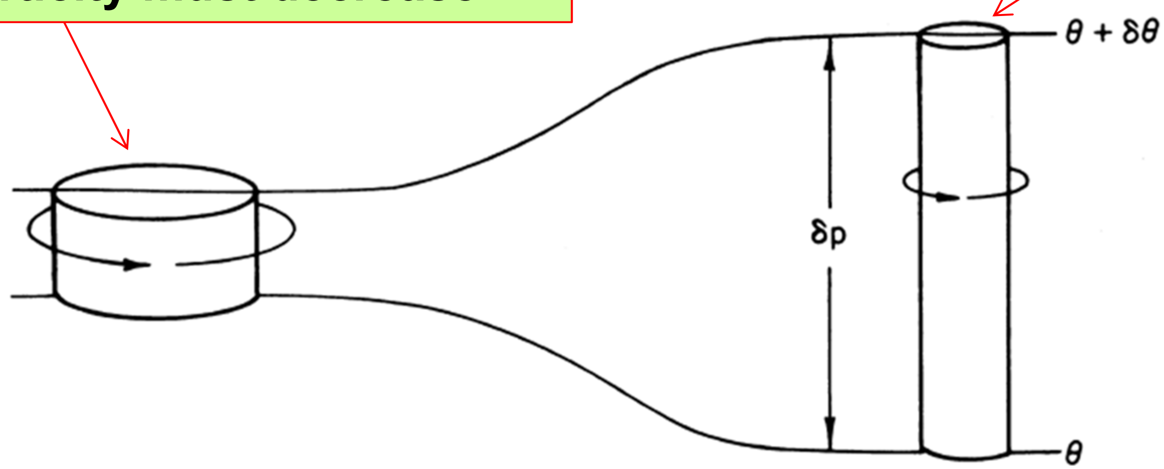
# *Ertel's Potential Vorticity*

$$P \equiv (\zeta_\theta + f)\left(-g\frac{\partial \theta}{\partial p}\right)$$

- PV is a product of absolute vorticity (the dynamic element) on an isentropic surface and static stability (the thermodynamic element).
- Values of potential vorticity are usually low in the troposphere (about 1 PVU) but increases rapidly to the stratosphere (about 4 PVU) due to the significant change of the static stability.

**static stability increases and absolute vorticity must decrease**

**static stability decreases and absolute vorticity must increase**



# *Why “Potential” Vorticity?*

$$P \equiv (\zeta_\theta + f)\left(-g\frac{\partial\theta}{\partial p}\right)$$

- There is the “potential” for generating vorticity by changing latitude (*f*) or changing static stability ( $\frac{\partial\theta}{\partial p}$ ).

- **Under constant stability, parcels moving south (north) will increase (decrease) in relative vorticity.**
- **Under constant relative vorticity, parcels moving south (north) will increase (decrease) in stability.**

# "Depth" of Potential Vorticity

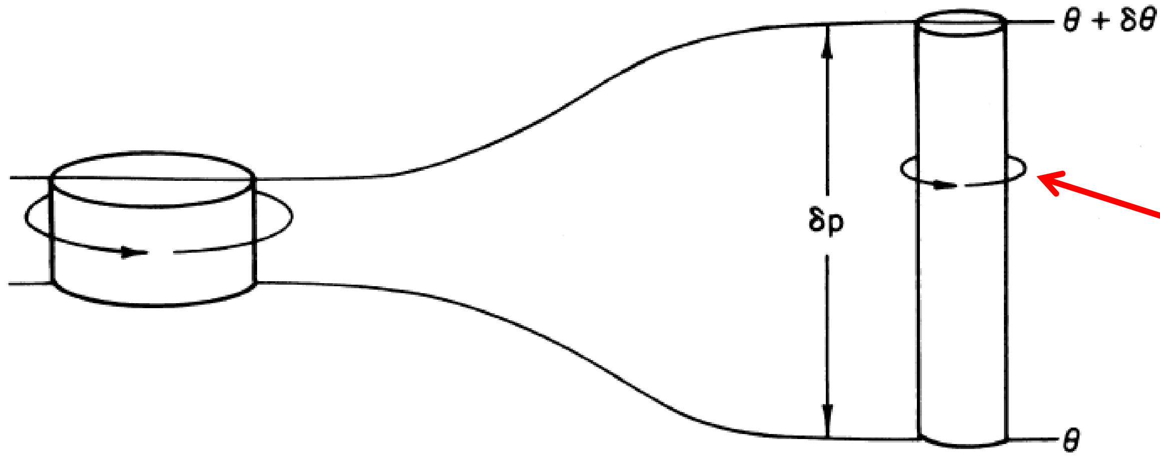



Fig. 4.7 A cylindrical column of air moving adiabatically, conserving potential vorticity.

"1/(depth)"

$$\frac{\partial \theta}{\partial p}$$

- In a homogeneous incompressible fluid, potential vorticity conservation takes a somewhat simpler form

Using $\delta A = M(\rho h)^{-1} = \mathrm{Const}/h$

$$(\zeta + f)/h = \eta/h = \mathrm{Const}$$

**Rossby Potential Vorticity Conservation Law**

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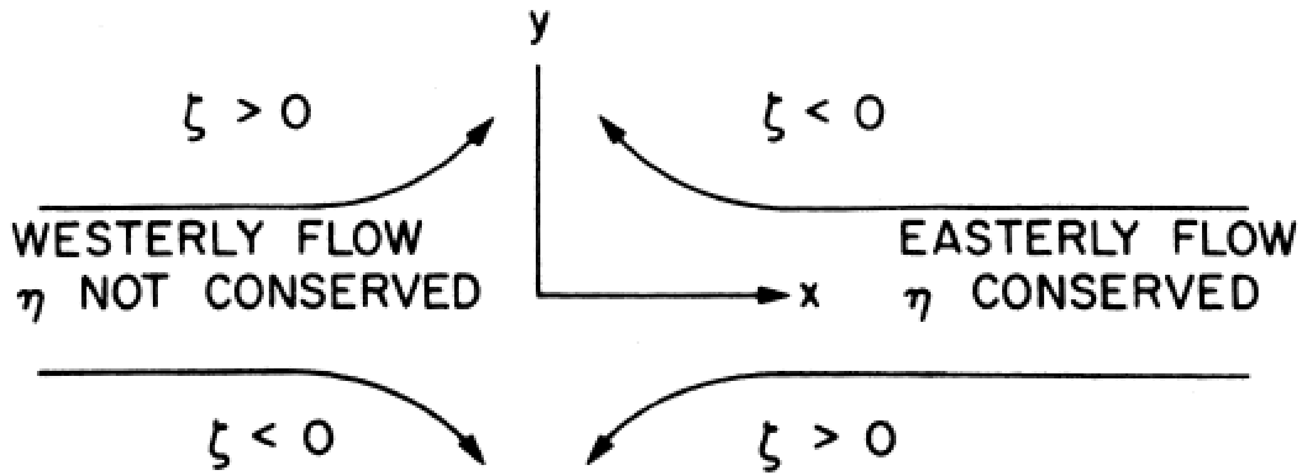
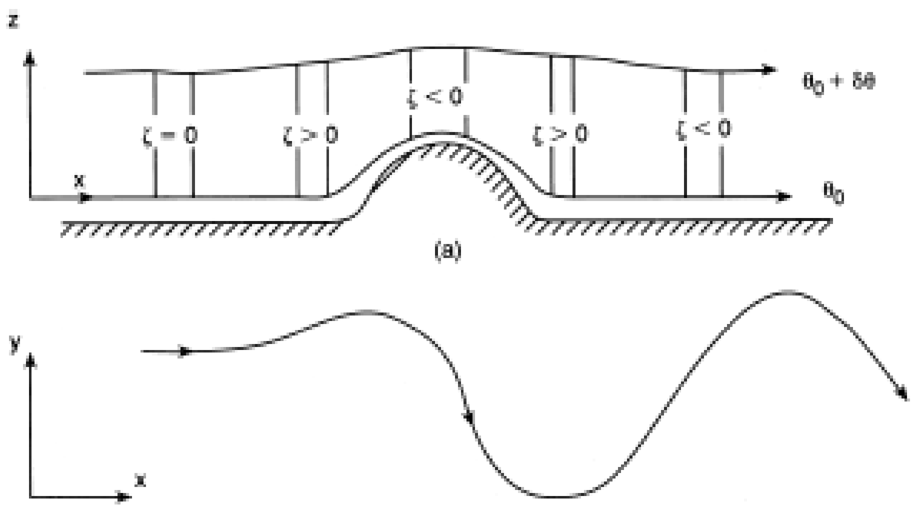



**Fig. 4.8** Absolute vorticity conservation for curved flow trajectories.

- Westerly zonal flow must remain purely zonal if absolute vorticity is to be conserved following the motion.
- Easterly current can curve either to the north or to the south and still conserve absolute vorticity.

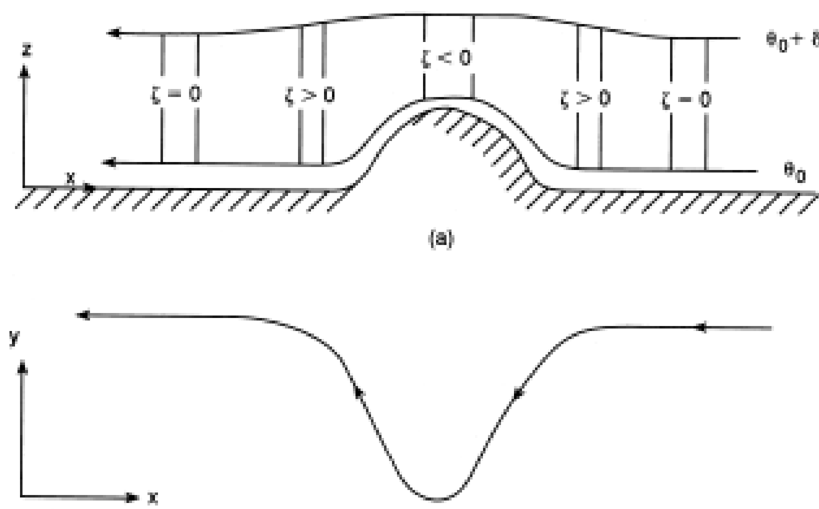
# Flows Cross Over a Mountain

**Westerly over mountain**



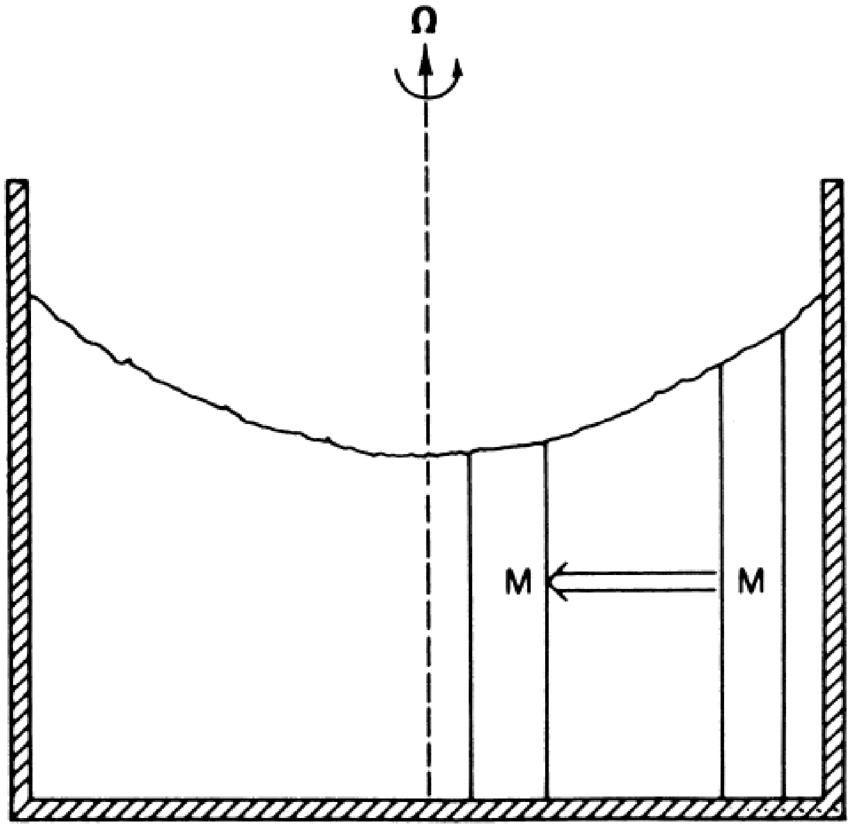

Steady westerly flow over a large-scale ridge will result in a cyclonic flow pattern immediately to
the east of the barrier (the lee side trough) followed by an alternating series of ridges and troughs downstream.

**Easterly over mountain**




In the case of an easterly wind, the disturbance in the streamlines damps out away from the barrier.

# Depth and Latitude




- The Rossby potential vorticity conservation law indicates that in a barotropic fluid, a change in the depth is dynamically analogous to a change in the Coriolis parameter.
- Therefore, in a barotropic fluid, a decrease of depth with increasing latitude has the same effect on the relative vorticity as the increase of the Coriolis force with latitude.

# Vorticity Equation

**(1) Begins with the Eq of motion:**

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}-fv=-\frac{1}{\rho}\frac{\partial p}{\partial x}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+fu=-\frac{1}{\rho}\frac{\partial p}{\partial y}\right)$$

**(2) Use the definition of relative vorticity (ζ):**

$$\frac{\partial \zeta}{\partial t}+u\frac{\partial \zeta}{\partial x}+v\frac{\partial \zeta}{\partial y}+w\frac{\partial \zeta}{\partial z}+(\zeta+f)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=$$
$$+\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)+v\frac{df}{dy}=\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y}-\frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)$$

**(3) We get the vorticity equation:**

*(1) divergence term*

$$\frac{D}{Dt}(\zeta+f)=-(\zeta+f)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$
$$-\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)+\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y}-\frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)$$

*(2) tilting term* *(3) solenoid term*

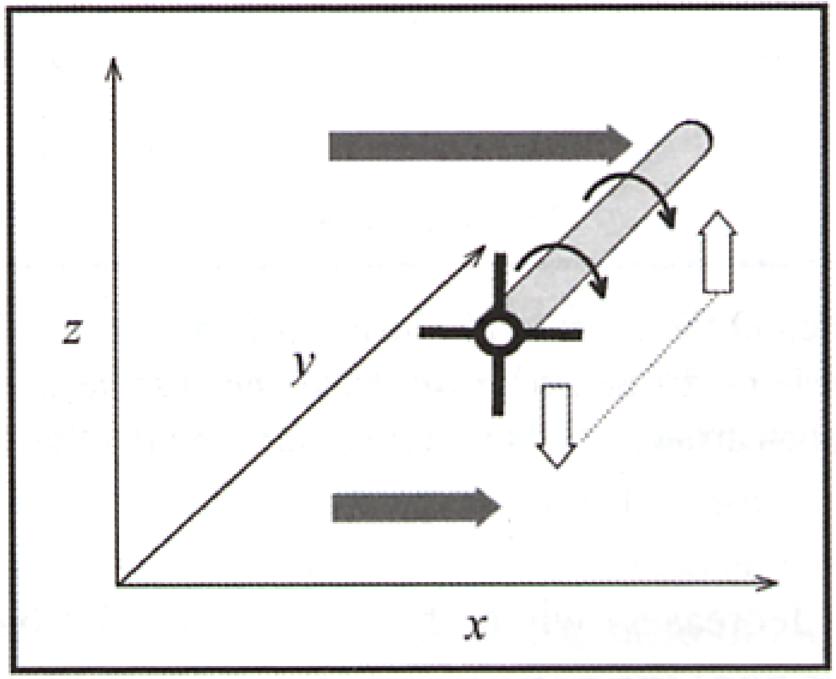
# Divergence Term

- If the horizontal flow is divergent, the area enclosed by a chain of fluid parcels will increase with time and if circulation is to be conserved, the average absolute vorticity of the enclosed fluid must decrease (i.e., the vorticity will be diluted).
- If, however, the flow is convergent, the area enclosed by a chain of fluid parcels will decrease with time and the vorticity will be concentrated.
- This mechanism for changing vorticity following the motion is very important in synoptic-scale disturbances.

# Tilting (or Twisting) Term

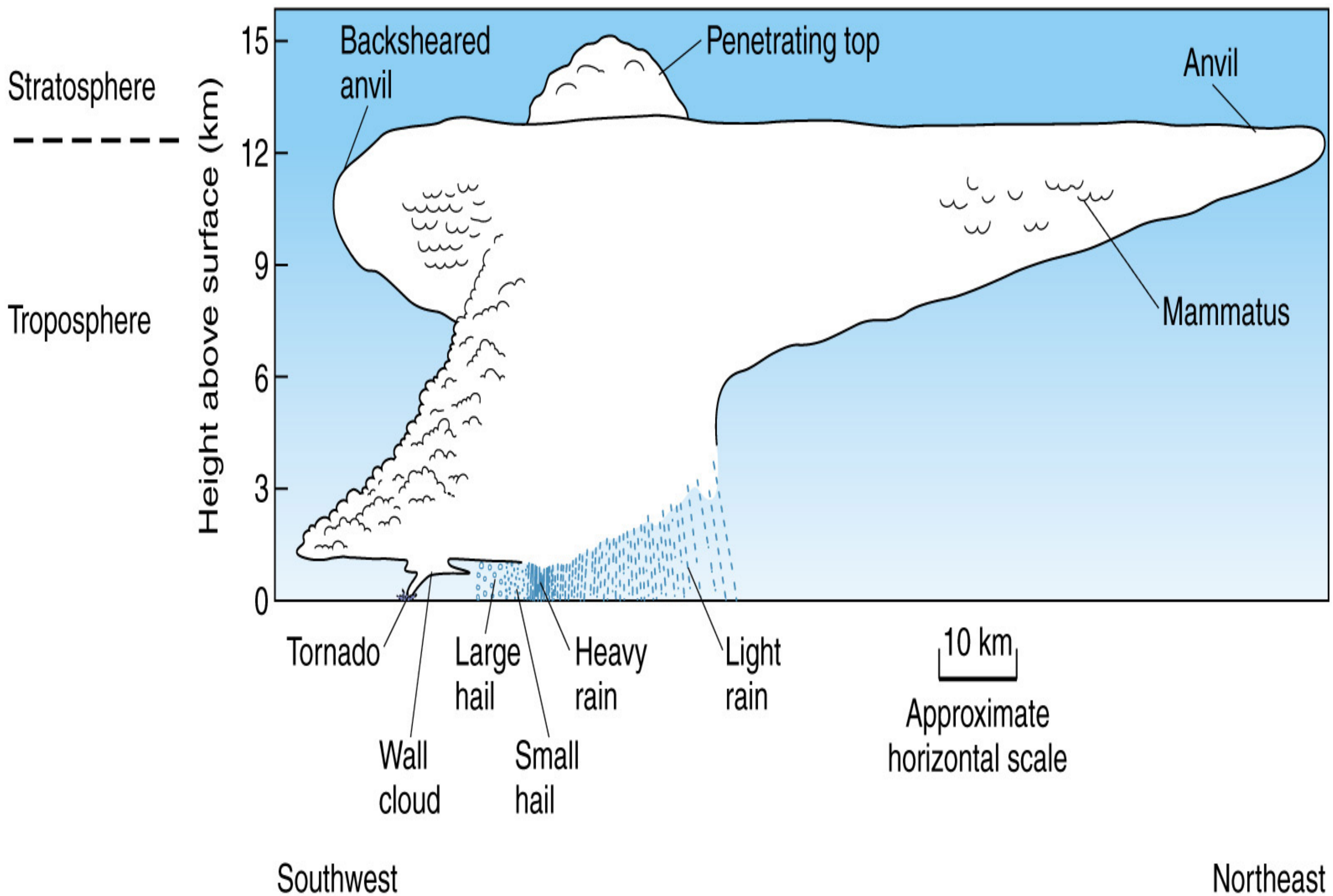
- Convert vorticity in X and Y directions into the Z-direction by the tilting/twisting effect produced by the vertical velocity (əw/əx and əw/əy).

$$-\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)$$

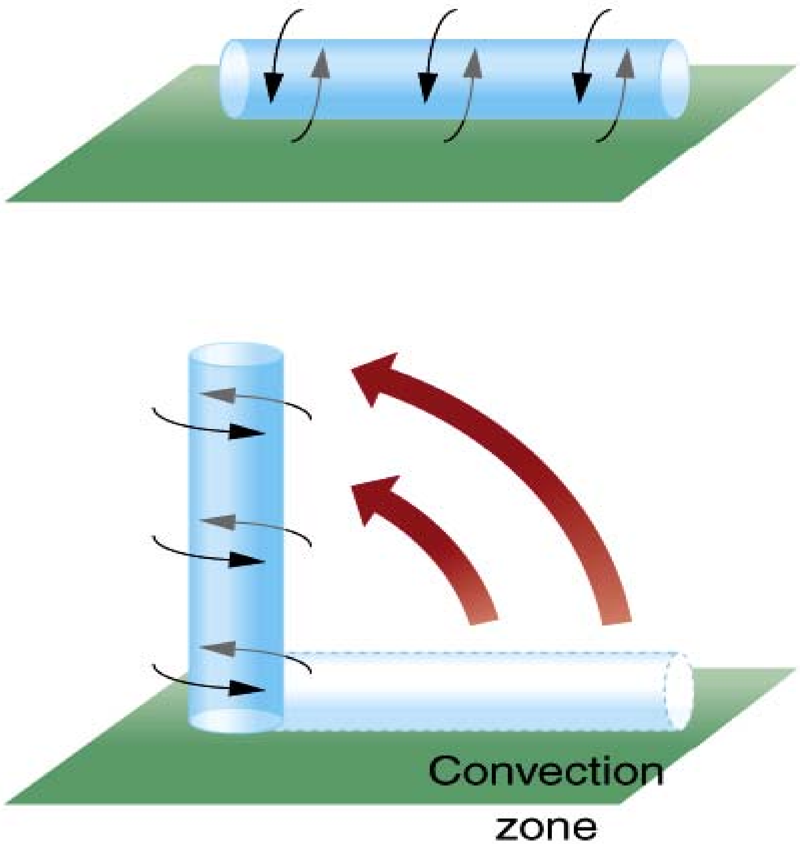



$\frac{\partial u}{\partial z}$ ➔ **vorticity in Y-direction**

$\frac{\partial w}{\partial y}$ ➔ **Tilting by the variation of w in Y-direction**

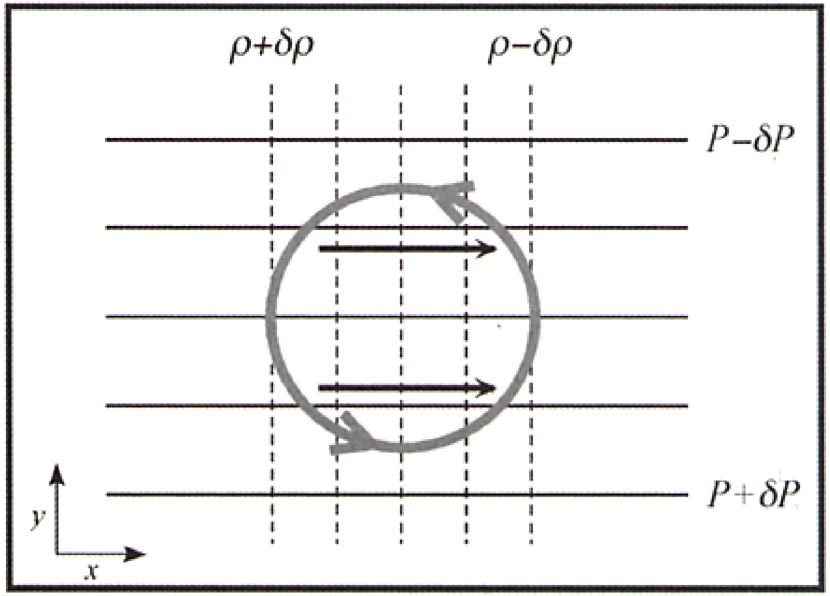

Stratosphere
Troposphere
Height above surface (km)
15
12
9
6
3
0
Backsheared anvil
Penetrating top
Anvil
Mammatus
Tornado
Wall cloud
Large hail
Small hail
Heavy rain
Light rain
10 km
Approximate horizontal scale
Southwest
Northeast

# Supercell Tornado Development




- Vertical wind shear creates a horizontal vortex.
- The vortex is tilted vertically by strong updrafts and forms a mesocyclone.
- The vortex stretches downward when the mesocyclone intensified.
- A wall cloud is formed under the cloud base, which then develops into a tornadoes.
- Only about 1/2 of all mesocyclones actually spawn a tornado

# Solenoid Term




$$\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y}-\frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)$$

- Given appropriate horizontal configurations of *p* and *ρ*, vorticity can be produced.
- In this example, cyclonic vorticity will rotate the iosteres until they are parallel with the isobars in a configuration in which high pressure corresponds to high density and vice versa.

# Scale Analysis of Vorticity Equation

$$\frac{D}{Dt}(\zeta + f) = -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)$$

**Scaled for mid-latitude synoptic-scale weather**

| | |
|---|---|
| $U \sim 10\,\mathrm{m\,s^{-1}}$ | horizontal scale |
| $W \sim 1\,\mathrm{cm\,s^{-1}}$ | vertical scale |
| $L \sim 10^6\,\mathrm{m}$ | length scale |
| $H \sim 10^4\,\mathrm{m}$ | depth scale |
| $\delta p \sim 10\,\mathrm{hPa}$ | horizontal pressure scale |
| $\rho \sim 1\,\mathrm{kg\,m^{-3}}$ | mean density |
| $\delta\rho/\rho \sim 10^{-2}$ | fractional density fluctuation |
| $L/U \sim 10^5\,\mathrm{s}$ | time scale |
| $f_0 \sim 10^{-4}\,\mathrm{s^{-1}}$ | Coriolis parameter |
| $\beta \sim 10^{-11}\,\mathrm{m^{-1}\,s^{-1}}$ | "beta" parameter |

$$\frac{D_h(\zeta + f)}{Dt} = -f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

$$\frac{D_h}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$$

# But for Intense Cyclonic Storms..

$$\frac{D_h(\zeta + f)}{Dt} = -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

- In intense cyclonic storms, the relative vorticity should be retained in the divergence term.

# For a Barotropic Flow
## (and incompressible)

(1) $$\frac{D_h(\zeta + f)}{Dt} = -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

➔ $$\frac{D_h(\zeta + f)}{Dt} = (\zeta + f)\left(\frac{\partial w}{\partial z}\right)$$

$$h\frac{D_h(\zeta_g + f)}{Dt} = (\zeta_g + f)\left[w(z_2) - w(z_1)\right] = \frac{Dz_2}{Dt} - \frac{Dz_1}{Dt} = \frac{D_h h}{Dt}$$

➔ $$\frac{1}{(\zeta_g + f)}\frac{D_h(\zeta_g + f)}{Dt} = \frac{1}{h}\frac{D_h h}{Dt}$$

➔ $$\frac{D_h \ln(\zeta_g + f)}{Dt} = \frac{D_h \ln h}{Dt}$$

➔ $$\frac{D_h}{Dt}\left(\frac{\zeta_g + f}{h}\right) = 0$$

**Rossby Potential Vorticity**

# Stream Function

- For horizontal motion that is non-divergent ($\partial u/\partial x + \partial v/\partial y = 0$), the flow field can be represented by a *streamfunction ψ (x, y) defined so that the* velocity components are given as

  $u = -\partial \psi/\partial y$,

  $v = +\partial \psi/\partial x$.

- The vorticity is then given by

$$\zeta = \partial v/\partial x - \partial u/\partial y = \partial^2 \psi/\partial x^2 + \partial^2 \psi/\partial y^2 \equiv \nabla^2 \psi$$

# Velocity Potential

A **velocity potential** is used in fluid dynamics, when a fluid occupies a simply-connected region and is irrotational. In such a case,

$$\nabla \times \mathbf{u} = 0,$$

where $\mathbf{u}$ denotes the flow velocity of the fluid. As a result, $\mathbf{u}$ can be represented as the gradient of a scalar function $\Phi$:

$$\mathbf{u} = \nabla \Phi,$$

$\Phi$ is known as a **velocity potential** for $\mathbf{u}$.