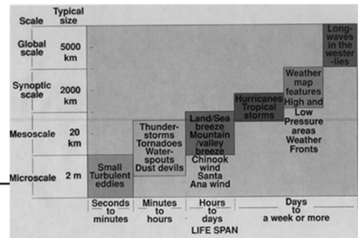
Lecture 2: Basic Conservation Laws

Type of motion	Horizontal scale (m)
Molecular mean free path	10^{-7}
Minute turbulent eddies	10^{-2} – 10^{-1}
Small eddies	10^{-1} – 1
Dust devils	1 – 10
Gusts	10 – 10^{2}
Tornadoes	10^{2}
Cumulonimbus clouds	10^{3}
Fronts, squall lines	10^{4} – 10^{5}
Hurricanes	10^{5}
Synoptic cyclones	10^{6}
Planetary waves	10^{7}



- Conservation of Momentum
- Conservation of Mass
- Conservation of Energy
- Scaling Analysis

Conservation Law of Momentum

$$\boxed{\frac{D_a \mathbf{U}_a}{Dt} = \sum \mathbf{F}}$$

Newton's 2nd Law of Momentum

$\mathbf{U}_a$ = absolute velocity viewed in an inertial system

$\frac{D_a \mathbf{U}_a}{Dt}$ = rate of change of ***Ua*** following the motion in an inertial system

- The conservation law for momentum (Newton's second law of motion) relates the rate of change of the absolute momentum following the motion in an inertial reference frame to the sum of the forces acting on the fluid.

Apply the Law to a Rotating Coordinate

- For most applications in meteorology it is desirable to refer the motion to a reference frame rotating with the earth.
- Transformation of the momentum equation to a rotating coordinate system requires a relationship between the total derivative of a vector in an inertial reference frame and the corresponding total derivative in a rotating system.

$$\frac{D_a \mathbf{U}_a}{Dt} \xrightarrow{\;?\;} \frac{D\mathbf{U}}{Dt}$$

The acceleration following the motion in an inertial coordinate

The acceleration following the motion in a rotating coordinate

Total Derivative in a Rotating Coordinate

$\mathbf{A} = \mathbf{i}' A'_x + \mathbf{j}' A'_y + \mathbf{k}' A'_z$ (in an inertial coordinate)

$\mathbf{A} = \mathbf{i} A_x + \mathbf{j} A_y + \mathbf{k} A_z$ (in a coordinate with an angular velocity $\mathbf{\Omega}$)

$$\begin{aligned}\frac{D_a \mathbf{A}}{Dt} &= \mathbf{i}' \frac{DA'_x}{Dt} + \mathbf{j}' \frac{DA'_y}{Dt} + \mathbf{k}' \frac{DA'_z}{Dt} \\ &= \underbrace{\mathbf{i}\frac{DA_x}{Dt} + \mathbf{j}\frac{DA_y}{Dt} + \mathbf{k}\frac{DA_z}{Dt}}_{\text{Change of vector A in the rotating coordinate}} + \underbrace{\frac{D_a \mathbf{i}}{Dt} A_x + \frac{D_a \mathbf{j}}{Dt} A_y + \frac{D_a \mathbf{k}}{Dt} A_z}_{\text{Change of the rotating coordinate view from the inertial coordinate}}\end{aligned}$$

$$\rightarrow \boxed{\frac{D_a \mathbf{A}}{Dt} = \frac{D\mathbf{A}}{Dt} + \mathbf{\Omega} \times \mathbf{A}}$$

Newton's 2nd Law in a Rotating Frame

$$\boxed{\frac{D_a \mathbf{U}_a}{Dt} = \sum \mathbf{F}}$$

using

$$\frac{D_a \mathbf{U}_a}{Dt} = \frac{D\mathbf{U}_a}{Dt} + \mathbf{\Omega} \times \mathbf{U}_a$$

covert acceleration from an inertial to a rotating frames

$$\mathbf{U}_a = \mathbf{U} + \mathbf{\Omega} \times \mathbf{r}$$

absolute velocity of an object on the rotating earth is equal to its velocity relative to the earth plus the velocity due to the rotation of the earth

$$\Rightarrow \frac{D_a \mathbf{U}_a}{Dt} = \frac{D}{Dt}(\mathbf{U} + \mathbf{\Omega} \times \mathbf{r}) + \mathbf{\Omega} \times (\mathbf{U} + \mathbf{\Omega} \times \mathbf{r})$$

[Here $\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) = \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{R}) = -\Omega^2 \mathbf{R}$]

$$\Rightarrow \frac{D_a \mathbf{U}_a}{Dt} = \frac{D\mathbf{U}}{Dt} + 2\mathbf{\Omega} \times \mathbf{U} - \Omega^2 \mathbf{R}$$

Coriolis force

Centrifugal force

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Momentum Conservation in a Rotating Frame

on an inertial coordinate

$$\frac{D_a \mathbf{U}_a}{Dt} = \sum \mathbf{F} = \text{pressure gradient force + true gravity + viscous force}$$

$$= -\frac{1}{\rho}\nabla p + \mathbf{g}^* + \mathbf{F}_r$$

Because $\frac{D_a \mathbf{U}_a}{Dt} = \frac{D\mathbf{U}}{Dt} + 2\mathbf{\Omega} \times \mathbf{U} - \Omega^2 \mathbf{R}$

$$\Rightarrow \frac{D\mathbf{U}}{Dt} + 2\mathbf{\Omega} \times \mathbf{U} - \Omega^2 \mathbf{R} = -\frac{1}{\rho}\nabla p + \mathbf{g}^* + \mathbf{F}_r$$

$$\Rightarrow \boxed{\frac{D\mathbf{U}}{Dt} = -2\mathbf{\Omega} \times \mathbf{U} - \frac{1}{\rho}\nabla p + \mathbf{g} + \mathbf{F}_r}$$

on a rotating coordinate

Here, g = apparent gravity = ($g^* + \Omega^2 R$)

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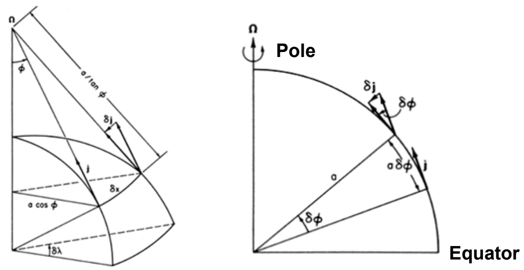
Momentum Equation on a Spherical Coordinate



$$\mathbf{U} \equiv \mathbf{i}u + \mathbf{j}v + \mathbf{k}w$$

$$u \equiv Dx/Dt = a\cos\phi\, D\lambda/Dt \equiv r\cos\phi \frac{D\lambda}{Dt},$$

$$v \equiv Dy/Dt = a\, D\phi/Dt \equiv r\frac{D\phi}{Dt},$$

$$w \equiv \frac{Dz}{Dt}$$

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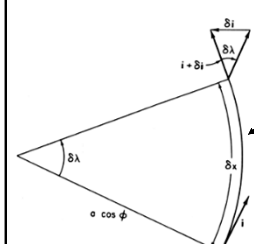
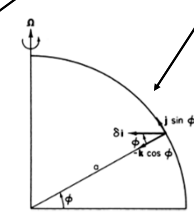
Rate of Change of *U*

$$\frac{D\mathbf{U}}{Dt} = \mathbf{i}\frac{Du}{Dt} + \mathbf{j}\frac{Dv}{Dt} + \mathbf{k}\frac{Dw}{Dt} + u\frac{D\mathbf{i}}{Dt} + v\frac{D\mathbf{j}}{Dt} + w\frac{D\mathbf{k}}{Dt}$$

$$\frac{D\mathbf{i}}{Dt} = u\frac{\partial \mathbf{i}}{\partial x} = \frac{u}{a\cos\phi}(\mathbf{j}\sin\phi - \mathbf{k}\cos\phi)$$





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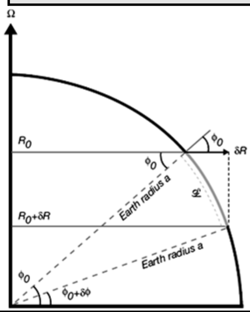
Coriolis Force (for n-s motion)

Suppose that an object of unit mass, initially at latitude φ moving zonally at speed u, relative to the surface of the earth, is displaced in latitude or in altitude by an impulsive force. As the object is displaced it will conserve its angular momentum in the absence of a torque in the east–west direction. Because the distance R to the axis of rotation changes for a displacement in latitude or altitude, the absolute angular velocity ($\Omega + u/R$) must change if the object is to conserve its absolute angular momentum. Here Ω is the angular speed of rotation of the earth. Because Ω is constant, the relative zonal velocity must change. ***Thus, the object behaves as though a zonally directed deflection force were acting on it***.



$$\left(\Omega + \frac{u}{R}\right) R^2 = \left(\Omega + \frac{u + \delta u}{R + \delta R}\right) (R + \delta R)^2$$

➔ $\delta u = -2\Omega \delta R - \frac{u}{R} \delta R$ (neglecting higher-orders)

using $\delta R = -\sin\phi \delta y$

➔ $$\boxed{\left(\frac{Du}{Dt}\right) = \left(2\Omega \sin\phi + \frac{u}{a} \tan\phi\right) \frac{Dy}{Dt} = 2\Omega v \sin\phi + \frac{uv}{a} \tan\phi}$$

and for a vertical displacement in which $\delta R = +\cos\phi \delta z$:

$$\boxed{\left(\frac{Du}{Dt}\right) = -\left(2\Omega \cos\phi + \frac{u}{a}\right) \frac{Dz}{Dt} = -2\Omega w \cos\phi - \frac{uw}{a}}$$

Momentum Eq. on Spherical Coordinate

$$\boxed{\frac{D\mathbf{U}}{Dt} = -2\mathbf{\Omega} \times \mathbf{U} - \frac{1}{\rho} \nabla p + \mathbf{g} + \mathbf{F}_r}$$

- $\frac{D\mathbf{U}}{Dt} = \left(\frac{Du}{Dt} - \frac{uv \tan\phi}{a} + \frac{uw}{a}\right) \mathbf{i} + \left(\frac{Dv}{Dt} + \frac{u^2 \tan\phi}{a} + \frac{vw}{a}\right) \mathbf{j} + \left(\frac{Dw}{Dt} - \frac{u^2 + v^2}{a}\right) \mathbf{k}$
- $-2\mathbf{\Omega} \times \mathbf{U} = -2\Omega \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \cos\phi & \sin\phi \\ u & v & w \end{vmatrix}$

  $= -(2\Omega w \cos\phi - 2\Omega v \sin\phi)\, \mathbf{i} - 2\Omega u \sin\phi\, \mathbf{j} + 2\Omega u \cos\phi\, \mathbf{k}$
- $\nabla p = \mathbf{i} \frac{\partial p}{\partial x} + \mathbf{j} \frac{\partial p}{\partial y} + \mathbf{k} \frac{\partial p}{\partial z}$
- $\mathbf{g} = -g\, \mathbf{k}$
- $\mathbf{F}_r = \mathbf{i} F_{rx} + \mathbf{j} F_{ry} + \mathbf{k} F_{rz}$

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Momentum Eq. on Spherical Coordinate

$$\frac{Du}{Dt} \boxed{- \frac{uv \tan\phi}{a} + \frac{uw}{a}} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + 2\Omega v \sin\phi - 2\Omega w \cos\phi + F_{rx}$$

$$\frac{Dv}{Dt} + \boxed{\frac{u^2 \tan\phi}{a} + \frac{vw}{a}} = -\frac{1}{\rho} \frac{\partial p}{\partial y} - 2\Omega u \sin\phi + F_{ry}$$

$$\frac{Dw}{Dt} \boxed{- \frac{u^2 + v^2}{a}} = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g + 2\Omega u \cos\phi + F_{rz}$$

Rate of change of the spherical coordinate

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Scaling Analysis

- Scale analysis, or scaling, is a convenient technique for estimating the magnitudes of various terms in the governing equations for a particular type of motion.
- In scaling, typical expected values of the following quantities are specified:

  (1) magnitudes of the field variables;

  (2) amplitudes of fluctuations in the field variables;

  (3) the characteristic length, depth, and time scales on which these fluctuations occur.
- These typical values are then used to compare the magnitudes of various terms in the governing equations.

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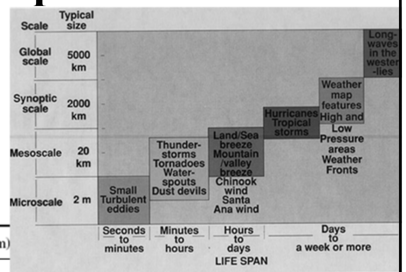
Scales of Atmospheric Motions



Type of motion	Horizontal scale (m)
Molecular mean free path	10^{-7}
Minute turbulent eddies	$10^{-2} - 10^{-1}$
Small eddies	$10^{-1} - 1$
Dust devils	$1 - 10$
Gusts	$10 - 10^2$
Tornadoes	10^2
Cumulonimbus clouds	10^3
Fronts, squall lines	$10^4 - 10^5$
Hurricanes	10^5
Synoptic cyclones	10^6
Planetary waves	10^7

Scaling for Synoptic-Scale Motion

- The complete set of the momentum equations describe all scales of atmospheric motions.

➔ We need to simplify the equation for synoptic-scale motions.

➔ We need to use the following characteristic scales of the field variables for mid-latitude synoptic systems:

$U \sim 10\ \text{m s}^{-1}$	horizontal velocity scale
$W \sim 1\ \text{cm s}^{-1}$	vertical velocity scale
$L \sim 10^6\ \text{m}$	length scale [$\sim 1/(2\pi)$ wavelength]
$H \sim 10^4\ \text{m}$	depth scale
$\delta P/\rho \sim 10^3\ \text{m}^2\ \text{s}^{-2}$	horizontal pressure fluctuation scale
$L/U \sim 10^5\ \text{s}$	time scale

Units of Atmospheric Pressure

- **Pascal (Pa):** a SI (Systeme Internationale) unit for air pressure.

  1 Pa = force of 1 newton acting on a surface of one square meter
  1 hectopascal (hPa) = 1 millibar (mb) [hecto = one hundred =100]

- **Bar:** a more popular unit for air pressure.

  1 bar = 1000 hPa = 1000 mb

- **One atmospheric pressure** = standard value of atmospheric pressure at lea level = 1013.25 mb = 1013.25 hPa.

Pressure Gradients

- Pressure Gradients
  – The pressure gradient force initiates movement of atmospheric mass, wind, from areas of higher to areas of lower pressure
- Horizontal Pressure Gradients
  – Typically only small gradients exist across large spatial scales (1mb/100km)
  – Smaller scale weather features, such as hurricanes and tornadoes, display larger pressure gradients across small areas (1mb/6km)
- Vertical Pressure Gradients
  – ***Average vertical pressure gradients are usually greater than extreme examples of horizontal pressure gradients*** as pressure always decreases with altitude (1mb/10m)

Scaling Results for the Horizontal Momentum Equations

	A	B	C	D	E	F	G
$x-$Eq.	$\frac{Du}{Dt}$	$-2\Omega v\sin\phi$	$+2\Omega w\cos\phi$	$+\frac{uw}{a}$	$-\frac{uv\tan\phi}{a}$	$=-\frac{1}{\rho}\frac{\partial p}{\partial x}$	$+F_{rx}$
$y-$Eq.	$\frac{Dv}{Dt}$	$+2\Omega u\sin\phi$		$+\frac{vw}{a}$	$+\frac{u^2\tan\phi}{a}$	$=-\frac{1}{\rho}\frac{\partial p}{\partial y}$	$+F_{ry}$
Scales	U^2/L	f_0U	f_0W	$\frac{UW}{a}$	$\frac{U^2}{a}$	$\frac{\delta P}{\rho L}$	$\frac{\nu U}{H^2}$
(m s^{-2})	10^{-4}	10^{-3}	10^{-6}	10^{-8}	10^{-5}	10^{-3}	10^{-12}

Geostrophic Approximation, Balance, Wind

Scaling for mid-latitude synoptic-scale motion

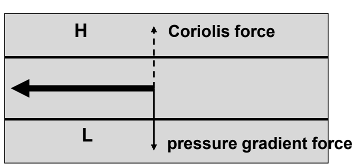
$$-fv \approx -\frac{1}{\rho}\frac{\partial p}{\partial x}; \qquad fu \approx -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\mathbf{V}_g \equiv \mathbf{k} \times \frac{1}{\rho f}\nabla p$$

Geostrophic wind

• The fact that the horizontal flow is in approximate geostrophic balance is helpful for diagnostic analysis.

Geostrophic Balance



❑ By doing scale analysis, it has been shown that large-scale and synoptic-scale weather system are in geostrophic balance.

❑ Geostrophic winds always follow the constant pressure lines (isobar). Therefore, we can figure out flow motion by looking at the pressure distribution.

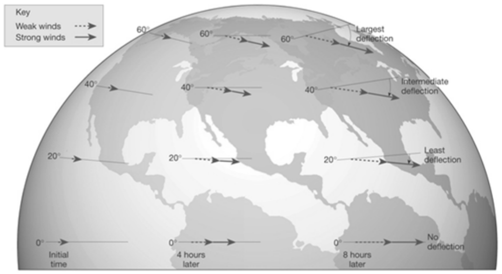
Weather Prediction

$$\frac{Du}{Dt} = fv - \frac{1}{\rho}\frac{\partial p}{\partial x} = f\left(v - v_g\right)$$

$$\frac{Dv}{Dt} = -fu - \frac{1}{\rho}\frac{\partial p}{\partial y} = -f\left(u - u_g\right)$$

- In order to obtain prediction equations, it is necessary to retain the acceleration term in the momentum equations.
- The geostrophic balance make the weather prognosis (prediction) difficult because acceleration is given by the small difference between two large terms.
- A small error in measurement of either velocity or pressure gradient will lead to very large errors in estimating the acceleration.

Coriolis Force Change with latitudes



(from *The Atmosphere*)

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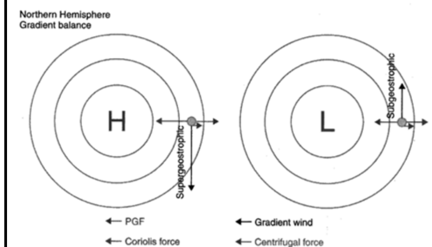
Super- and Sub-Geostrophic Wind



(from *Meteorology: Understanding the Atmosphere*)

- For high pressure system
- → gradient wind > geostrophic wind
- → supergeostropic.
- For low pressure system
- → gradient wind < geostrophic wind
- → subgeostropic.

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Gradient Wind Balance

- The three-way balance of horizontal pressure gradient, Coriolis force, and the centrifugal force is call the ***gradient wind balance***.
- The gradient wind is an excellent approximation to the actual wind observed <u>*above*</u> the Earth's surface, especially at the middle latitudes.

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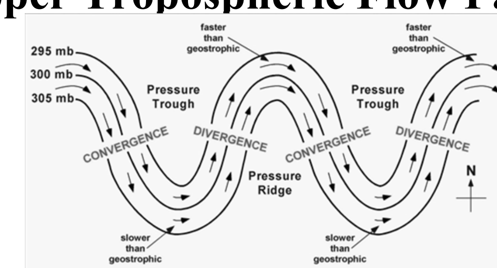
Upper Tropospheric Flow Pattern



- Upper tropospheric flows are characterized by trough (low pressure; isobars dip southward) and ridge (high pressure; isobars bulge northward).
- The winds are in gradient wind balance at the bases of the trough and ridge and are slower and faster, respectively, than the geostrophic winds.
- Therefore, convergence and divergence are created at different parts of the flow patterns, which contribute to the development of the low and high systems.

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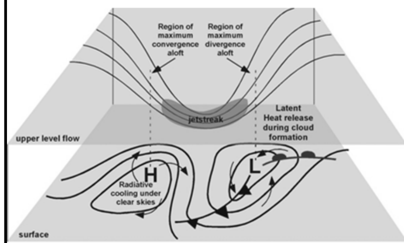
Developments of Low- and High-Pressure Centers



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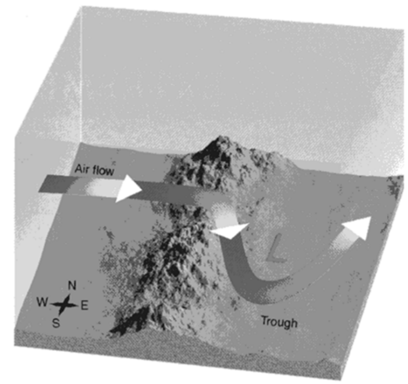
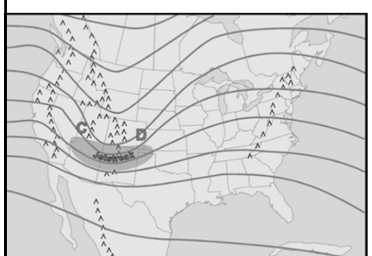
- ***Dynamic Effects***: Combined curvature and jetstreak effects produce upper-level convergence on the west side of the trough to the north of the jetsreak, which add air mass into the vertical air column and tend to produce a surface high-pressure center. The same combined effects produce a upper-level divergence on the east side of the trough and favors the formation of a low-level low-pressure center.
- ***Thermodynamic Effect***: heating ➔ surface low pressure; cooling ➔ surface high pressure.
- ***Frictional Effect***: Surface friction will cause convergence into the surface low-pressure center after it is produced by upper-level dynamic effects, which adds air mass into the low center to "fill" and weaken the low center (increase the pressure)
- ***Low Pressure***: The evolution of a low center depends on the relative strengths of the upper-level development and low-level friction damping.
- ***High Pressure***: The development of a high center is controlled more by the convergence of surface cooling than by the upper-level dynamic effects. Surface friction again tends to destroy the surface high center.

Lee-Side Lows



Initial Development of the Cyclone



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- Cyclones form east of the Rockies when a wave and/or a jetstreak move across the Rockies from the west at jetstream level.
- A cyclone appears as a center of low pressure on surface chart.
- Low pressure forms when divergence occurs aloft associated with embedded jetstreaks and changes in curvature within the jetstream.

•As soon as low pressure begins to develop, air in the lower troposphere will start to circulate about the low-pressure center. A cyclone has begun to form.

Extratropical Cyclones in North America

Cyclones preferentially form in five locations in North America:

(1) East of the Rocky Mountains
(2) East of Canadian Rockies
(3) Gulf Coast of the US
(4) East Coast of the US
(5) Bering Sea & Gulf of Alaska

Rossby Number

$$\boxed{R_0 \equiv U/(f_0 L)}$$

- Rossby number is a non-dimensional measure of the magnitude of the acceleration compared to the Coriolis force:
- The smaller the Rossby number, the better the geostrophic balance can be used.

$$(U^2/L)/(f_0 U)$$

- Rossby number measure the relative importnace of the inertial term and the Coriolis term.
- This number is about O(0.1) for Synoptic weather and about O(1) for ocean.

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Scaling Analysis for Vertical Momentum Eq.

z - Eq.	Dw/Dt	$-2\Omega u\cos\phi$	$-(u^2+v^2)/a$	$= -\rho^{-1}\partial p/\partial z$	$-g$	$+F_{rz}$
Scales	UW/L	f_0U	U^2/a	$P_0/(\rho H)$	g	νWH^{-2}
m s^{-2}	10^{-7}	10^{-3}	10^{-5}	10	10	10^{-15}

$$\boxed{\frac{Dw}{Dt} - \frac{u^2+v^2}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + 2\Omega u\cos\phi + F_{rz}}$$

Hydrostatic Balance

$$\boxed{\frac{1}{\rho_0}\frac{dp_0}{dz} \equiv -g}$$

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Hydrostatic Balance

z - Eq.	Dw/Dt	$-2\Omega u\cos\phi$	$-(u^2+v^2)/a$	$= -\rho^{-1}\partial p/\partial z$	$-g$	$+F_{rz}$
Scales	UW/L	f_0U	U^2/a	$P_0/(\rho H)$	g	νWH^{-2}
m s^{-2}	10^{-7}	10^{-3}	10^{-5}	10	10	10^{-15}

- The acceleration term is several orders smaller than the hydrostatic balance terms.

➔ Therefore, for synoptic scale motions, vertical accelerations are negligible and the vertical velocity cannot be determined from the vertical momentum equation.

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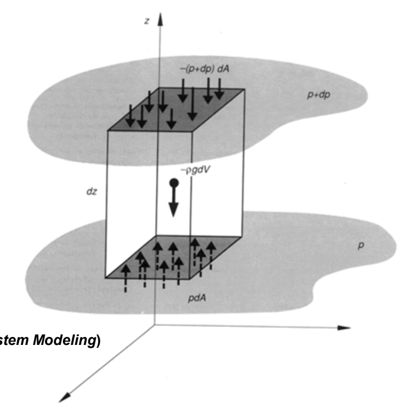
Hydrostatic Balance in the Vertical



(from *Climate System Modeling*)

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Vertical Motions

- For synoptic-scale motions, the vertical velocity component is typically of the order of a few centimeters per second. Routine meteorological soundings, however, only give the wind speed to an accuracy of about a meter per second.
- Thus, in general the vertical velocity is not measured directly but must be inferred from the fields that are measured directly.
- Two commonly used methods for inferring the vertical motion field are (1) the ***kinematic method***, based on the equation of continuity, and (2) the ***adiabatic method***, based on the thermodynamic energy equation.

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Primitive Equations

$$\frac{du}{dt} - fv = -\frac{\partial \Phi}{\partial x}$$ **(1) zonal momentum equation**

$$\frac{dv}{dt} + fu = -\frac{\partial \Phi}{\partial y}$$ **(2) meridional momentum equation**

$$\frac{dp}{dz} = -\rho g$$ **(3) hydrostatic equation**

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0$$ **(4) continuity equation**

$$c_p \frac{dT}{dt} - \alpha \frac{dp}{dt} = Q$$ **(5) thermodynamic energy equation**

$$p = \rho RT$$ **(6) equation of state**

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The Kinematic Method

- We can integrate the continuity equation in the vertical to get the vertical velocity.

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)_p + \frac{\partial \omega}{\partial p} = 0$$

$$\Rightarrow \quad \omega(p) = \omega(p_s) - \int_{p_s}^{p} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)_p dp$$

- We use the information of horizontal divergence to infer the vertical velocity. However, for midlatitude weather, the horizontal divergence is due primarily to the small departures of the wind from geostrophic balance. A 10% error in evaluating one of the wind components can easily cause the estimated divergence to be in error by 100%.
- For this reason, the continuity equation method is not recommended for estimating the vertical motion field from observed horizontal winds.

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The Adiabatic Method

- The adiabatic method is not so sensitive to errors in the measured horizontal velocities, is based on the thermodynamic energy equation.

$$\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) - S_p\omega = \frac{J}{c_p}$$

$$\Rightarrow \quad \omega = S_p^{-1}\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right)$$

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Conservation of Mass

- The mathematical relationship that expresses conservation of mass for a fluid is called the *continuity equation*.

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \mathbf{U}) = 0}$$ (mass divergence form)

$$\boxed{\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla \bullet \mathbf{U} = 0}$$ (velocity divergence form)

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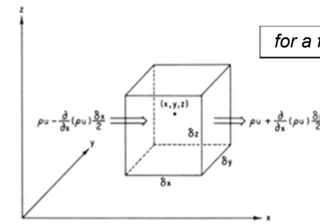
Mass Divergence Form (Eulerian View)



- Net rate of mass inflow through the sides = Rate of mass accumulation within the volume
- Net flow through the δy• δ z surface)

$$\left[\rho u - \frac{\partial}{\partial x}(\rho u)\frac{\delta x}{2}\right]\delta y \delta z - \left[\rho u + \frac{\partial}{\partial x}(\rho u)\frac{\delta x}{2}\right]\delta y \delta z = -\frac{\partial}{\partial x}(\rho u)\delta x \delta y \delta z$$

- Net flow from all three directions = $-\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right]\delta x \delta y \delta z$
- Rate of mass accumulatio $\frac{\partial \rho}{\partial t}\delta x \delta y \delta z$
- Mass conservation $\boxed{\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \mathbf{U}) = 0}$

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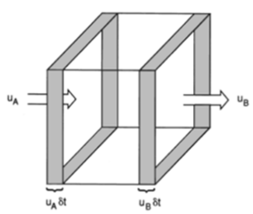
Velocity Divergence Form (Lagragian View)



$$\delta u = u_B - u_A = D(x + \delta x)/Dt - Dx/Dt$$
$$\delta u = D(\delta x)/Dt$$

- Following a control volume of a fixed mass (δ M), the amount of mass is

$$\frac{1}{\delta M}\frac{D}{Dt}(\delta M) = \frac{1}{\rho \delta V}\frac{D}{Dt}(\rho \delta V) = \frac{1}{\rho}\frac{D\rho}{Dt} + \frac{1}{\delta V}\frac{D}{Dt}(\delta V) = 0$$

- $\frac{1}{\delta V}\frac{D}{Dt}(\delta V) = \frac{1}{\delta x}\frac{D}{Dt}(\delta x) + \frac{1}{\delta y}\frac{D}{Dt}(\delta y) + \frac{1}{\delta z}\frac{D}{Dt}(\delta z)$ and $\begin{cases} \delta u = D(\delta x)/Dt \\ \delta v = D(\delta y)/Dt \\ \delta w = D(\delta z)/D. \end{cases}$

$$\lim_{\delta x, \delta y, \delta z \to 0}\left[\frac{1}{\delta V}\frac{D}{Dt}(\delta V)\right] = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla \bullet \mathbf{U}$$

$$\boxed{\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla \bullet \mathbf{U} = 0}$$

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Scaling Analysis of Continuity Eq.

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla \bullet \mathbf{U} = 0$$

$$\rightarrow \underbrace{\frac{1}{\rho_0}\left(\frac{\partial \rho'}{\partial t} + \mathbf{U} \bullet \nabla \rho'\right)}_{A} + \underbrace{\frac{w}{\rho_0}\frac{d\rho_0}{dz}}_{B} + \underbrace{\nabla \bullet \mathbf{U}}_{C} \approx 0$$

where ρ' designates the local deviation of density from its horizontally averaged value, $\rho_0(z)$. For synoptic scale motions $\rho'/\rho_0 \sim 10^{-2}$

Term A ➔ $\frac{1}{\rho_0}\left(\frac{\partial \rho'}{\partial t} + \mathbf{U} \bullet \nabla \rho'\right) \sim \frac{\rho'}{\rho_0}\frac{U}{L} \approx 10^{-7}\mathrm{s}^{-1}$

Term B ➔ $\frac{w}{\rho_0}\frac{d\rho_0}{dz} \backsim \frac{W}{H} \approx 10^{-6}\mathrm{s}^{-1}$ (because $d \ln \rho_0 / dz \sim H^{-1}$)

Term C ➔ $\nabla \bullet \mathbf{U} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \backsim 10^{-1}\frac{U}{L} \approx 10^{-6}\mathrm{s}^{-1}$ and $\frac{\partial w}{\partial z} \sim \frac{W}{H} \approx 10^{-6}\mathrm{s}^{-1}$

$$\rightarrow \boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} + w\frac{d}{dz}(\ln \rho_0) = 0 \quad \text{or} \quad \nabla \bullet (\rho_0 \mathbf{U}) = 0}$$

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Meaning of the Scaled Continuity Eq.

$$\boxed{\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla \bullet \mathbf{U} = 0} \xrightarrow{\text{scaled}} \boxed{\nabla \bullet (\rho_0 \mathbf{U}) = 0}$$

- Velocity divergence vanishes ($\nabla \bullet \mathbf{U} = 0$) in an incompressible fluid.

- For purely horizontal flow, the atmosphere behaves as though it were an incompressible fluid.
- However, when there is vertical motion the compressibility associated with the height dependence of ρ0 must be taken into account.

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The First Law of Thermodynamics

- This law states that (1) heat is a form of energy that (2) its conversion into other forms of energy is such that total energy is conserved.
- The change in the internal energy of a system is equal to the heat added to the system minus the work down by the system:

$$\boxed{\Delta U = Q - W}$$

change in internal energy (related to temperature) **Heat added to the system** **Work done by the system**

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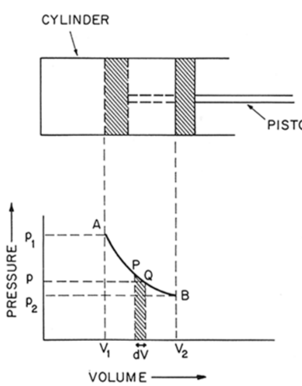
(from *Atmospheric Sciences: An Intro. Survey)*



- Therefore, when heat is added to a gas, there will be some combination of an expansion of the gas (i.e. the work) and an increase in its temperature (i.e. the increase in internal energy):

Heat added to the gas = work done by the gas + temp. increase of the gas

$$\Delta H = p\,\Delta\alpha + C_v\,\Delta T$$

volume change of the gas　specific heat at constant volume

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Heat and Temperature

- Heat and temperature are both related to the internal kinetic energy of air molecules, and therefore can be related to each other in the following way:

$$\boxed{Q = c*m*\Delta T}$$

Heat added **Mass** **Temperature changed**

Specific heat = the amount of heat per unit mass required to raise the temperature by one degree Celsius

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Specific Heat

TABLE 2.1 The Specific Heat of a Substance is the Amount of Heat Required to Increase the Temperature of One Gram of the Substance 1° C

	Specific Heat	
Substance	*(cal/g/°C)*	*(J/kg/°C)*
Water	1.0	4186
Ice	0.50	2093
Air	0.24	1005
Sand	0.19	795

(from *Meteorology: Understanding the Atmosphere*)

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Apply the Energy Conservation to a Control Volume

- The first law of thermodynamics is usually derived by considering a system in thermodynamic equilibrium, that is, a system that is initially at rest and after exchanging heat with its surroundings and doing work on the surroundings is again at rest.
- A Lagrangian control volume consisting of a specified mass of fluid may be regarded as a thermodynamic system. However, unless the fluid is at rest, it will not be in thermodynamic equilibrium. Nevertheless, the first law of thermodynamics still applies.
- The ***thermodynamic energy*** of the control volume is considered to consist of the ***sum of the internal energy*** (due to the kinetic energy of the individual molecules) ***and the kinetic energy*** due to the macroscopic motion of the fluid. The rate of change of this total thermodynamic energy is equal to the rate of diabatic heating plus the rate at which work is done on the fluid parcel by external forces.

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Total Thermodynamic Energy

- If we let e designate the internal energy per unit mass, then the total thermodynamic energy contained in a Lagrangian fluid element of density ρ and volume δV is

$$\rho[e + (1/2)\mathbf{U} \bullet \mathbf{U}]\, \delta V$$

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External Forces

- The external forces that act on a fluid element may be divided into surface forces, such as pressure and viscosity, and body forces, such as gravity or the Coriolis force.
- However, because the Coriolis force is perpendicular to the velocity vector, it can do no work.

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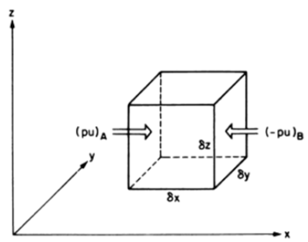
Work done by Pressure



The rate at which the surrounding fluid does work on the element due to the pressure force on the two boundary surfaces in the y, z plane is given by:

$$\boxed{(pu)_A \delta y \delta z - (pu)_B \delta y \delta z}$$

$$(pu)_B = (pu)_A + \left[\frac{\partial}{\partial x}(pu)\right]_A \delta x + \ldots$$

So the net rate at which the pressure force does work due to the x component of motion is →

$$[(pu)_A - (pu)_B]\delta y \delta z = -\left[\frac{\partial}{\partial x}(pu)\right]_A \delta V$$

Hence, the total rate at which work is done by the pressure force is simply →

$$\boxed{-\nabla \bullet (p\mathbf{U})\delta V}$$

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Thermodynamic Eq. for a Control Volume

$$\frac{D}{Dt}\left[\rho\left(e + \frac{1}{2}\mathbf{U} \bullet \mathbf{U}\right)\delta V\right] = -\nabla \bullet (p\mathbf{U})\delta V + \rho \mathbf{g} \bullet \mathbf{U}\delta V + \rho J \delta V$$

Work done by pressure force

Work done by gravity force

J is the rate of heating per unit mass due to radiation, conduction, and latent heat release.

(effects of molecular viscosity are neglected)

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Final Form of the Thermodynamic Eq.

$$\boxed{c_v \frac{DT}{Dt} + p\frac{D\alpha}{Dt} = J}$$

- After many derivations, this is the usual form of the thermodynamic energy equation.
- The second term on the left, representing the rate of working by the fluid system (per unit mass), represents a conversion between thermal and mechanical energy.
- This conversion process enables the solar heat energy to drive the motions of the atmosphere.

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Entropy Form of Energy Eq.

$$c_v \frac{DT}{Dt} + p\frac{D\alpha}{Dt} = J$$

P α = RT

Cp=Cv+R

$$c_p \frac{DT}{Dt} - \alpha\frac{Dp}{Dt} = J$$

Divided by T

$$c_p \frac{D\ln T}{Dt} - R\frac{D\ln p}{Dt} = \frac{J}{T} \equiv \frac{Ds}{Dt}$$

- The rate of change of entropy (s) per unit mass following the motion for a thermodynamically *reversible* process.
- A reversible process is one in which a system changes its thermodynamic state and then returns to the original state without changing its surroundings.

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Potential Temperature (θ)

- For an ideal gas undergoing an *adiabatic* process (i.e., a reversible process in which no heat is exchanged with the surroundings; $J=0$), the first law of thermodynamics can be written in differential form as:

$$c_p D\ln T - RD\ln p = D\left(c_p \ln T - R\ln p\right) = 0$$

$$\rightarrow \boxed{\theta = T\,(p_s/p)^{R/c_p}} \qquad \rightarrow \boxed{c_p \frac{D\ln\theta}{Dt} = \frac{J}{T} = \frac{Ds}{Dt}}$$

- Thus, every air parcel has a unique value of potential temperature, and this value is conserved for dry adiabatic motion.
- Because synoptic scale motions are approximately adiabatic outside regions of active precipitation, θ is a quasi-conserved quantity for such motions.
- Thus, for reversible processes, fractional potential temperature changes are indeed proportional to entropy changes.
- A parcel that conserves entropy following the motion must move along an isentropic (constant θ) surface.

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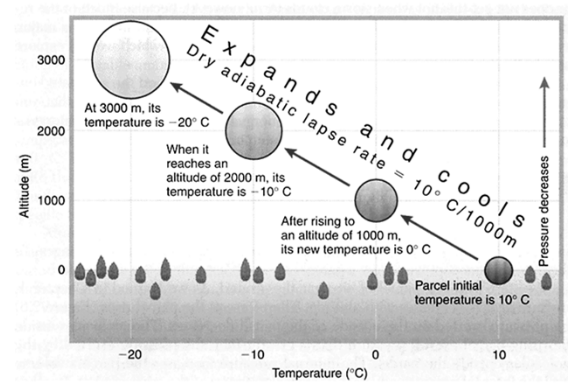
Dry Adiabatic Lapse Rate



(from *Meteorology: Understanding the Atmosphere*)

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Static Stability

If potential temperature is a function of height, the atmospheric lapse rate, $\Gamma \equiv -\partial T/\partial z$, will differ from the adiabatic lapse rate and

$$\frac{T}{\theta}\frac{\partial\theta}{\partial z} = \Gamma_d - \Gamma$$

If $\Gamma < \Gamma d$ so that θ increases with height, an air parcel that undergoes an adiabatic displacement from its equilibrium level will be positively buoyant when displaced downward and negatively buoyant when displaced upward so that it will tend to return to its equilibrium level and the atmosphere is said to be statically stable or *stably stratified*.

$$\begin{aligned} d\theta_0/dz &> 0 \quad &&\text{statically stable,}\\ d\theta_0/dz &= 0 \quad &&\text{statically neutral,}\\ d\theta_0/dz &< 0 \quad &&\text{statically unstable.}\end{aligned}$$

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Scaling of the Thermodynamic Eq.

$$C_p\frac{dT}{dt} - \alpha\frac{dp}{dt} = J$$

$$\rightarrow C_p\frac{dT}{dt} = J + \alpha\frac{\partial p}{\partial t} + \alpha(U\cdot\nabla p) + w\frac{\partial p}{\partial z}$$

$$\rightarrow C_p\frac{dT}{dt} = J + \boxed{\alpha\frac{\partial p}{\partial t} + \alpha(U\cdot\nabla p)} - wg$$

Small terms; neglected after scaling

$$\rightarrow \frac{dT}{dt} = \frac{J}{C_p} - \frac{g}{C_p}w$$

$$\rightarrow \frac{\partial T}{\partial t} = \frac{J}{C_p} - \frac{g}{C_p}w - U\cdot\nabla T - w\frac{\partial T}{\partial z}$$

$$\rightarrow \frac{\partial T}{\partial t} = \frac{J}{C_p} - V\cdot\nabla T - w\left(\frac{g}{C_p} + \frac{\partial T}{\partial z}\right)$$

$$\rightarrow \boxed{\frac{\partial T}{\partial t} = \frac{J}{C_p} - V\cdot\nabla T - w(\Gamma_d - \Gamma)}$$

$\Gamma = -\partial T/\partial z$ = lapse rate

$\Gamma_d = g/c_p$ = dry lapse rate

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Pressure Gradients

- Pressure Gradients
  - The pressure gradient force initiates movement of atmospheric mass, wind, from areas of higher to areas of lower pressure
- Horizontal Pressure Gradients
  - Typically only small gradients exist across large spatial scales (1mb/100km)
  - Smaller scale weather features, such as hurricanes and tornadoes, display larger pressure gradients across small areas (1mb/6km)
- Vertical Pressure Gradients
  - ***Average vertical pressure gradients are usually greater than extreme examples of horizontal pressure gradients*** as pressure always decreases with altitude (1mb/10m)

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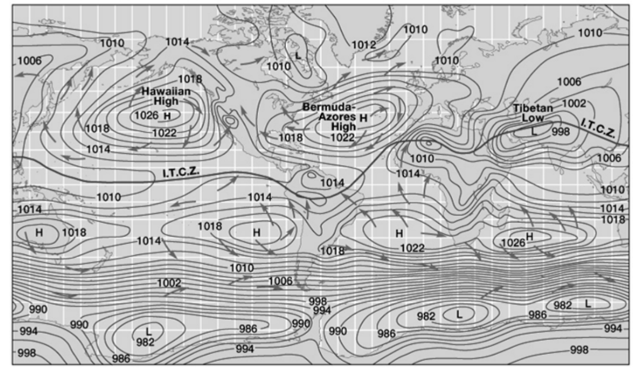

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Temperature Tendency Equation

$$\boxed{\frac{\partial T}{\partial t} = \frac{J}{C_p} - V \cdot \nabla T - w(\Gamma_d - \Gamma)}$$

Term A **Term B** **Term C**

- Term A: Diabatic Heating
- Term B: Horizontal Advection
- Term C: Adiabatic Effects

(heating/cooling due to vertical motion in a stable/unstable atmosphere)

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Primitive Equations

- The scaling analyses results in a set of approximate equations that describe the conservation of momentum, mass, and energy for the atmosphere.
- These sets of equations are called the primitive equations, which are very close to the original equations are used for numerical weather prediction.
- The primitive equations does not describe the moist process and are for a dry atmosphere.

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Primitive Equations

$$\frac{du}{dt} - fv = -\frac{\partial \Phi}{\partial x}$$ (1) zonal momentum equation

$$\frac{dv}{dt} + fu = -\frac{\partial \Phi}{\partial y}$$ (2) meridional momentum equation

$$\frac{dp}{dz} = -\rho g$$ (3) hydrostatic equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0$$ (4) continuity equation

$$c_p \frac{dT}{dt} - \alpha \frac{dp}{dt} = Q$$ (5) thermodynamic energy equation

$$p = \rho R T$$ (6) equation of state

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