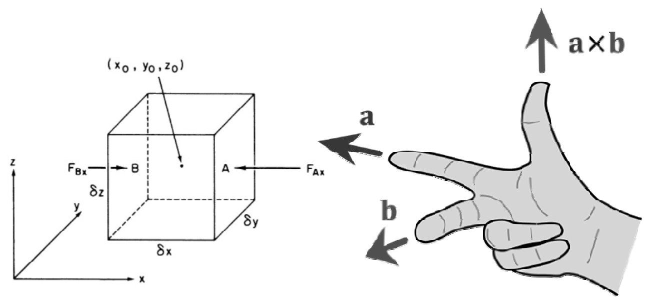
Lecture 1: Introduction and Review



- Review of fundamental mathematical tools
- Fundamental and apparent forces

Dynamics and Kinematics

- Kinematics: The term **kinematics** means *motion*. Kinematics is the study of motion without regard for the cause.
- Dynamics: On the other hand, **dynamics** is the study of the *causes* of motion.

This course discusses the physical laws that govern atmosphere/ocean motions.

Basic Conservation Laws

Atmospheric motions are governed by three fundamental physical principles:

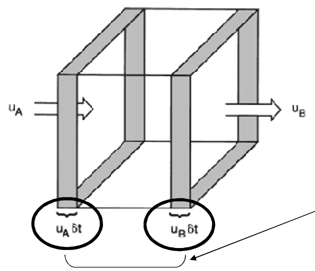
- conservation of mass (continuity equation)
- conservation of momentum (Newton's 2nd law of motion)
- conservation of energy (1st law of thermodynamics)

We need to develop mathematical formulas to describe these basic laws.

Control Volume

- The mathematical relations that express these laws may be derived by considering the budgets of mass, momentum, and energy for an infinitesimal *control volume* in the fluid.
- Two types of control volume are commonly used in fluid dynamics: *Eulerian* and *Lagrangian*.

Lagrangian View of Control Volume



Change in Lagrangian control volume (shown by shading) due to fluid motion parallel to the x axis.

- In the *Lagrangian* frame, the control volume consists of an infinitesimal mass of "tagged" fluid particles.
- The control volume moves about following the motion of the fluid, always containing the same fluid particles.

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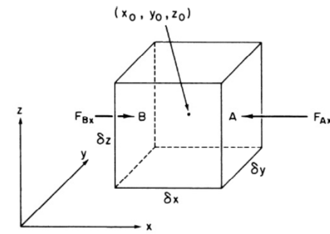
Eulerian View of Control Volume



- In the *Eulerian* frame of reference the control volume consists of a parallelepiped of sides δx, δy, δz, whose position is fixed relative to the coordinate axes.
- Mass, momentum, and energy budgets will depend on fluxes caused by the flow of fluid through the boundaries of the control volume.

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Linking Lagrangian and Eulerian Views

- The conservation laws to be derived contain expressions for the rates of change of density, momentum, and thermodynamic energy following the motion of particular fluid parcels.

➔ The Lagrangian frame is particularly useful for deriving conservation laws.

- However, observations are usually taken at fixed locations.

➔ The conservation laws are often applied in the Eulerian frame.

Therefore, it is necessary to derive a relationship between the rate of change of a field variable following the motion (i.e., total derivative; Lagrangian view) and its rate of change at a fixed point (i.e., local derivative; Eulerian view).

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Eulerian and Lagrangian Views

- **Eulerian view of the flow field** is a way of looking at fluid motion that focuses on specific locations in the space through which the fluid flows.
- **Lagrangian view of the flow field** is a way of looking at fluid motion where the observer follows an individual fluid parcel as it moves through space and time.

In order to apply conservation laws in the Eulerian frame, it is necessary to derive the relationship between the rate of change of a field variable **following the motion** and its rate of change at **a fixed location**.

local derivative $\frac{\partial F}{\partial t}$

total derivative $\frac{dF}{dt}$

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Linking Total Derivative to Local Derivative

In order to relate the total derivative to the local rate of change at a fixed point, we consider the temperature measured on a balloon that moves with the wind. Suppose that this temperature is T_0 at the point x_0, y_0, z_0 and time t_0. If the balloon moves to the point $x_0+\delta x$, $y_0+\delta y$, $z_0+\delta z$ in a time increment δt, then the temperature change recorded on the balloon, δT, can be expressed in a Taylor series expansion as

$$\delta T = \left(\frac{\partial T}{\partial t}\right)\delta t + \left(\frac{\partial T}{\partial x}\right)\delta x + \left(\frac{\partial T}{\partial y}\right)\delta y + \left(\frac{\partial T}{\partial z}\right)\delta z + (\text{ higher order terms})$$

Dividing through by δt and noting that δT is the change in temperature following the motion so that

$$\frac{DT}{Dt} \equiv \lim_{\delta t \to 0} \frac{\delta T}{\delta t}$$

we find that in the limit $\delta t \to 0$

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \left(\frac{\partial T}{\partial x}\right)\frac{Dx}{Dt} + \left(\frac{\partial T}{\partial y}\right)\frac{Dy}{Dt} + \left(\frac{\partial T}{\partial z}\right)\frac{Dz}{Dt}$$

is the rate of change of T following the motion.

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Taylor Series Expansion

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots$$

- **Taylor series** is a representation of a function as an infinite sum of terms calculated from the values of its derivatives at a single point.
- It is common practice to use a finite number of terms of the series to approximate a function.

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Partial Differential

$$\frac{\partial}{\partial a_i} f(\mathbf{a}) = \lim_{h \to 0} \frac{f(a_1, \ldots, a_{i-1}, a_i + h, a_{i+1}, \ldots, a_n) - f(a_1, \ldots, a_n)}{h}$$

- A **partial derivative** of a function of several variables is its derivative with respect to one of those variables, with the others held constant.
- As opposed to the **total derivative**, in which all variables are allowed to vary.

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If we now let

$$\frac{Dx}{Dt} \equiv u, \quad \frac{Dy}{Dt} \equiv v, \quad \frac{Dz}{Dt} \equiv w$$

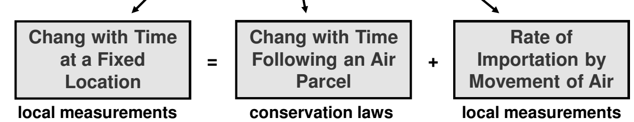
then u, v, w are the velocity components in the x, y, z directions, respectively, and

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}\right)$$

Using vector notation this expression may be rewritten as

$$\frac{\partial T}{\partial t} = \frac{DT}{Dt} - \mathbf{U} \cdot \nabla T$$

where $\mathbf{U} = \mathbf{i}u + \mathbf{j}v + \mathbf{k}w$ is the velocity vector. The term $-\mathbf{U} \cdot \nabla T$ is called the temperature *advection*. It gives the contribution to the local temperature change due to air motion.



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Example

Q: The surface pressure decreases by 3 hPa per 180 km in the eastward direction. A ship steaming eastward at 10 km/h measures a pressure fall of 1 hPa per 3 h. What is the pressure change on an island that the ship is passing?

A: The pressure change on the island ($\frac{\partial p}{\partial t}$) can be linked to the pressure change on the ship ($\frac{Dp}{Dt}$) in the following way:

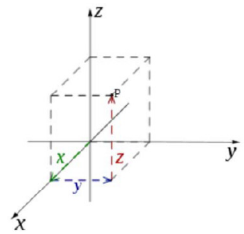
$$\frac{\partial p}{\partial t} = \frac{Dp}{Dt} - u\frac{\partial p}{\partial x}$$

$$\Rightarrow \quad \frac{\partial p}{\partial t} = \frac{-1\ \text{hPa}}{3\ \text{h}} - \left(10\frac{\text{km}}{\text{h}}\right)\left(\frac{-3\ \text{hPa}}{180\ \text{km}}\right) = -\frac{1\ \text{hPa}}{6\ \text{h}}$$

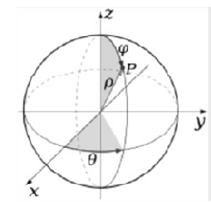
Coordinate System

A coordinate system is needed to describe the location in space.

(1) Cartesian (x, y, z)

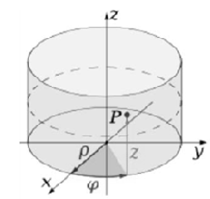


(2) Spherical (ρ, φ, θ)



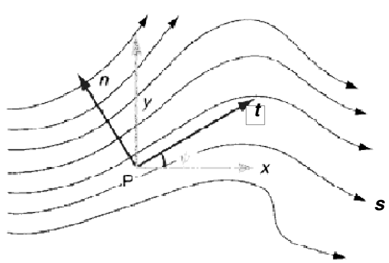
The spherical coordinate system, showing a point, P and its coordinates, ρ, θ and φ

(3) Cylindrical (ρ, φ, z)



A cylindrical coordinate system, showing radius, ρ, azimuth, φ and height, z.

Natural Coordinate



$$\mathbf{V} = V\,\mathbf{t}$$

$$V \equiv Ds/Dt$$

- At any point on a horizontal surface, we can define a pair of a system of natural coordinates (***t, n***), where ***t*** is the length directed downstream along the local streamline, and ***n*** is distance directed normal to the streamline and toward the left.

State Variable

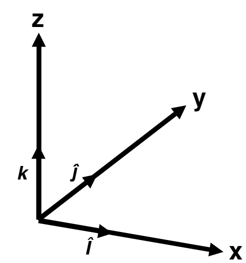
- Fundamental state variables (A) in the atmosphere (such as temperature, pressure, moisture, geopotential height, and 3-D wind) are function of the independent variables of space (x, y, z) and time (t):

$$A = A\,(x, y, z, t)$$

Scalar and Vector

- Many physical quantities in the atmosphere are described entirely in terms of *magnitude*, known as ***scalars*** (such as pressure and temperature).
- There are other physical quantities (such as 3D-wind or gradient of scalar) are characterized by both *magnitude* and *direction,* such quantities are known as ***vectors***.
- Any description of the fluid atmosphere contains reference to both ***scalars*** and ***vectors***.
- The mathematical descriptions of these quantities are known as ***vector analysis***.

Representation of Vector



$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

Vector Multiplication

(1) Multiplication by a scalar

$$c\vec{A} = (cA_x)\hat{i} + (cA_y)\hat{j} + (cA_z)\hat{k}$$

(2) Dot product (scalar product) ➔ scalar (e.g., advection)

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

(3) Cross product (vector product) ➔ vector (e.g., vorticity)

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = (A_y B_z - A_z B_y)\hat{i} + (A_z B_x - A_x B_z)\hat{j} + (A_x B_y - A_y B_x)\hat{k}$$

Dot Product

Cross Product

Four Most Important Vector Operations

Operation	Notation	Description	Domain/Range
Gradient	$\mathrm{grad}(f) = \nabla f$	Measures the rate and direction of change in a scalar field.	Maps scalar fields to vector fields.
Curl	$\mathrm{curl}(\mathbf{F}) = \nabla \times \mathbf{F}$	Measures the tendency to rotate about a point in a vector field.	Maps vector fields to vector fields.
Divergence	$\mathrm{div}(\mathbf{F}) = \nabla \cdot \mathbf{F}$	Measures the magnitude of a source or sink at a given point in a vector field.	Maps vector fields to scalar fields.
Laplacian	$\Delta f = \nabla^2 f = \nabla \cdot \nabla f$	A composition of the divergence and gradient operations.	Maps scalar fields to scalar fields.

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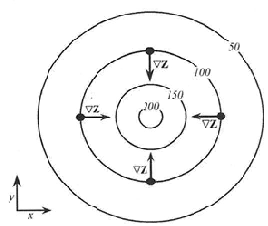
Gradient (Derivative) Operator

$$\nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}}$$



- We will often need to describe both the magnitude and direction of the derivative of a scalar field, by employing a mathematical operator known as the ***del operator***.

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Curl (Rotor) Operator

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\mathbf{k}$$

- The **curl** (or **rotor**) is a vector operator that describes the rotation of a vector field.
- At every point in the field, the curl is represented by a vector.
- The length and direction of the vector characterize the rotation at that point.
- The curl is a form of differentiation for vector fields.
- A vector field whose curl is zero is called irrotational.

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Example

- In this case, the curl is actually a constant, irrespective of position.
- Using the right-hand rule, we expect the curl to be into the page.

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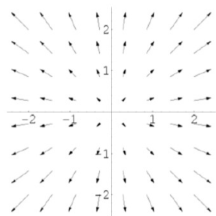
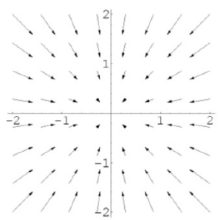
Divergence Operator

$$\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z}.$$

$\nabla\cdot\mathbf{F} > 0$





$\nabla\cdot\mathbf{F} < 0$

- **divergence** is an operator that measures the magnitude of a vector field's source or sink at a given point, in terms of a signed scalar.
- Negative values of divergence is also known as "**convergence**".

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Laplacian Operator

$$\nabla^2 f = \nabla\cdot\nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

- The **Laplace operator** is used in the modeling of wave propagation, heat flow, and fluid mechanics.

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Fundamental and Apparent Forces

- Newton's second law of motion states that the rate of change of momentum (i.e., the acceleration) of an object, as measured relative to coordinates fixed in space, equals the sum of all the forces acting.
- For atmospheric motions of meteorological interest, the forces that are of primary concern are the pressure gradient force, the gravitational force, and friction. These are the *fundamental* forces.
- For a coordinate system rotating with the earth, Newton's second law may still be applied provided that certain *apparent* forces, the centrifugal force and the Coriolis force, are included among the forces acting.

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Inertial and Noninertial Reference Frames

- In formulating the laws of atmospheric dynamics it is natural to use a *geocentric* reference frame, that is, a frame of reference at rest with respect to the rotating earth.
- Newton's first law of motion states that a mass in uniform motion relative to a coordinate system fixed in space will remain in uniform motion in the absence of any forces.
- Such motion is referred to as *inertial motion*; and the fixed reference frame is an inertial, or absolute, frame of reference.
- It is clear, however, that an object at rest or in uniform motion with respect to the rotating earth is not at rest or in uniform motion relative to a coordinate system fixed in space.
- Therefore, motion that appears to be inertial motion to an observer in a geocentric reference frame is really accelerated motion.
- Hence, a geocentric reference frame is a *noninertial* reference frame.
- Newton's laws of motion can only be applied in such a frame if the acceleration of the coordinates is taken into account.
- The most satisfactory way of including the effects of coordinate acceleration is to introduce "apparent" forces in the statement of Newton's second law.
- These apparent forces are the inertial reaction terms that arise because of the coordinate acceleration.
- For a coordinate system in uniform rotation, two such apparent forces are required: the centrifugal force and the Coriolis force.

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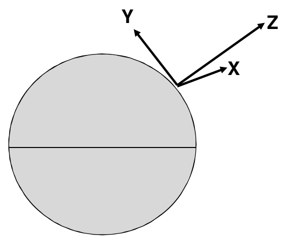
Convention of Using Cartesian Coordinate



- X increases toward the east.
- Y increases toward the north.
- Z is zero at surface of earth and increases upward.

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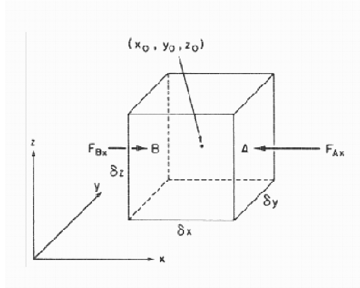
Pressure Gradient Force



Use Taylor expansion ➔ $F_{Ax} = -\left(p_0 + \frac{\partial p}{\partial x}\frac{\delta x}{2}\right)\delta y\,\delta z$

$F_{Bx} = +\left(p_0 - \frac{\partial p}{\partial x}\frac{\delta x}{2}\right)\delta y\,\delta z$

$F_x = F_{Ax} + F_{Bx} = -\frac{\partial p}{\partial x}\delta x\,\delta y\,\delta z$

$m = \rho\delta x\delta y\delta z$ ← *Mass of the air parcel*

$\frac{F_x}{m} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$ ← *Force per unit mass*

$\frac{F_y}{m} = -\frac{1}{\rho}\frac{\partial p}{\partial y}$ and $\frac{F_z}{m} = -\frac{1}{\rho}\frac{\partial p}{\partial z}$

Pressure Gradient Force ➔ $\frac{\mathbf{F}}{m} = -\frac{1}{\rho}\nabla p$

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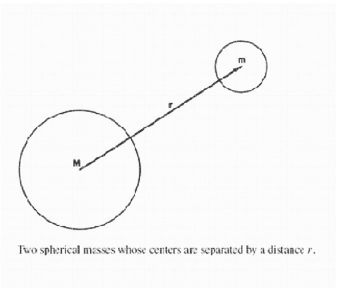
Gravitational Force

- Newton's law of universal gravitation states that any two elements of mass in the universe attract each other with a force proportional to their masses and inversely proportional to the square of the distance separating them. '
- Thus, if the earth is designated as mass M and m is a mass element of the atmosphere, then the force per unit mass exerted on the atmosphere by the gravitational attraction of the earth is



Two spherical masses whose centers are separated by a distance r.

$$\frac{\mathbf{F}_g}{m} \equiv \mathbf{g}^* = -\frac{GM}{r^2}\left(\frac{\mathbf{r}}{r}\right)$$

$r = a + z$: (a: earth radius; z: height above surface)

$$\mathbf{g}^* = \frac{\mathbf{g}_0^*}{(1 + z/a)^2}$$

where $\mathbf{g}_0^* = -(GM/a^2)(\mathbf{r}/r)$ is the gravitational force at mean sea level.

For meteorological applications,

$z \ll a$ ➔ $\mathbf{g}^* = \mathbf{g}_0^*$

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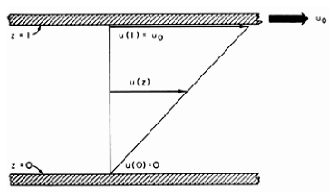
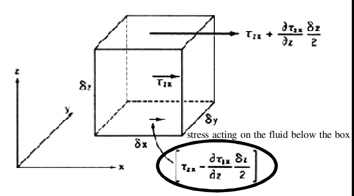
Frictional (Viscous) Force





- Any real fluid is subject to internal friction (viscosity)
- Force required to maintain this flow
  ➔ $F = \mu A u_0 / l$
- For a layer of fluid at depth δZ, the force is
  ➔ $F = \mu A \delta u / \delta z$
- Viscous force per unit area (shearing stress):
  ➔ $\tau_{zx} = \lim_{\delta z \to 0} \mu \frac{\delta u}{\delta z} = \mu \frac{\partial u}{\partial z}$

- Stresses applied on a fluid element
  ➔ $\left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\frac{\delta z}{2}\right)\delta x\,\delta y - \left(\tau_{zx} - \frac{\partial \tau_{zx}}{\partial z}\frac{\delta z}{2}\right)\delta x\,\delta y$
  stress acting on the box from the fluid below
- Viscous force per unit mass due to stress
  ➔ $\frac{1}{\rho}\frac{\partial \tau_{zx}}{\partial z} = \frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right)$
- Frictional force per unit mass in x-direction
  ➔ $F_{rx} = \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right]$
  where $\nu = \mu/\rho$

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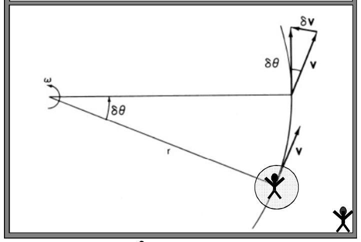
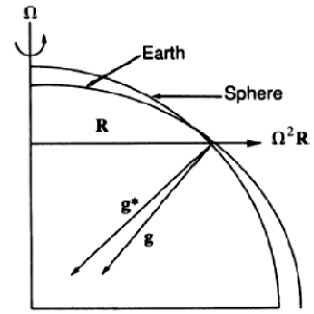
Centrifugal Force

A ball of mass *m* is attached to a string and whirled through a circle of radius *r* at a constant angular velocity ω.



- **An observer ( ) in an inertial coordinate**
  - → sees the ball's direction keeps on changing
  - → there is a centripetal force applied to the ball
- **Acceleration of the ball =** (δV / δt)
  - → $|\delta \mathbf{V}| = |\mathbf{V}|\,\delta\theta$ and in the limit of $\delta t \rightarrow 0$.
  - → $\frac{D\mathbf{V}}{Dt} = |\mathbf{V}|\frac{D\theta}{Dt}\left(-\frac{\mathbf{r}}{r}\right)$ ("**-**": force goes inward)
  - → because $|\mathbf{V}| = \omega r$ and $D\theta / Dt = \omega$.
  - → $\frac{D\mathbf{V}}{Dt} = -\omega^2 r$ = **-(V^2/r)**
  - → The ball is pulled inward by the centripetal force
- **An observer ( ) rotating with the ball**
  - → sees no change in the ball's motion
  - → but the observer sees the pulling of the ball by the string
  - → in order to apply Newton's 2nd law to describe the ball's motion in this rotation coordinate
  - → an apparent force has to be added which is opposite to centripetal force
  - → centrifugal force = - centripetal force = $\omega^2 r$

Apparent Gravity (g) and Geopotential (Φ)



Relationship between the true gravitation vector g* and gravity g. For an idealized homogeneous spherical earth, g* would be directed toward the center of the earth. In reality, g* does not point exactly to the center except at the equator and the poles. Gravity, g, is the vector sum of g* and the centrifugal force and is perpendicular to the level surface of the earth which approximates an oblate spheroid.

- An object at rest on earth's surface experiences both the gravitational force (g*) and a centrifugal force ($-\Omega^2$R).
- As a result, these two forces together result in an "apparent gravity" force (g):

  $$\mathbf{g} \equiv -g\mathbf{k} \equiv \mathbf{g}^* + \Omega^2\mathbf{R}$$

- True gravity (g) is perpendicular to a sphere, but the apparent gravity (g*) is perpendicular to earth's surface.
- Geopotential is the work required to raised a unit mass to height z from mean sea level:

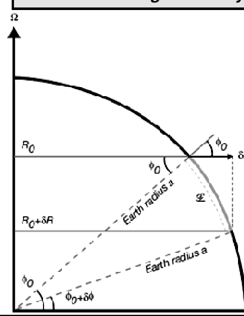
  $$\Phi = \int_0^z g\,dz \quad \text{and} \quad \nabla\Phi = -\mathbf{g}$$

Coriolis Force

- By adding the "apparent" centrifugal force, we can use Newton's 2nd law of motion to describe the force balance for an object ***at rest*** on the surface of the earth.
- We need to add an additional "apparent" Coriolis force in the 2nd law if the object is ***in motion*** with respect to the surface of the earth.

Coriolis Force (for n-s motion)

Suppose that an object of unit mass, initially at latitude φ moving zonally at speed u, relative to the surface of the earth, is displaced in latitude or in altitude by an impulsive force. As the object is displaced it will conserve its angular momentum in the absence of a torque in the east–west direction. Because the distance *R* to the axis of rotation changes for a displacement in latitude or altitude, the absolute angular velocity ($\Omega + u/R$) must change if the object is to conserve its absolute angular momentum. Here Ω is the angular speed of rotation of the earth. Because Ω is constant, the relative zonal velocity must change. ***Thus, the object behaves as though a zonally directed deflection force were acting on it***.



$$\left(\Omega + \frac{u}{R}\right)R^2 = \left(\Omega + \frac{u+\delta u}{R+\delta R}\right)(R+\delta R)^2$$

→ $\delta u = -2\Omega\delta R - \frac{u}{R}\delta R$ (neglecting higher-orders)

using $\delta R = -\sin\phi\,\delta y$

→ $$\left(\frac{Du}{Dt}\right) = \left(2\Omega\sin\phi + \frac{u}{a}\tan\phi\right)\frac{Dy}{Dt} = 2\Omega v\sin\phi + \frac{uv}{a}\tan\phi$$

and for a vertical displacement in which $\delta R = +\cos\phi\,\delta z$:

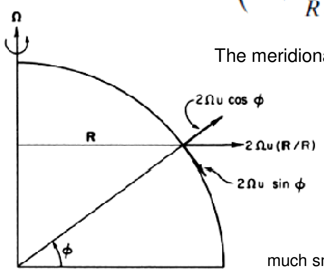
$$\left(\frac{Du}{Dt}\right) = -\left(2\Omega\cos\phi + \frac{u}{a}\right)\frac{Dz}{Dt} = -2\Omega w\cos\phi - \frac{uw}{a}$$

Coriolis Force (for e-w motion)

Suppose now that the object is set in motion in the eastward direction by an impulsive force. Because the object is now rotating faster than the earth, the centrifugal force on the object will be increased.

The excess of the centrifugal force over that for an object at rest becomes the Coriolis force for this zonal motion on a rotation coordinate:

$$\left(\Omega + \frac{u}{R}\right)^2 \mathbf{R} - \Omega^2 \mathbf{R} = \frac{2\Omega u \mathbf{R}}{R} + \frac{u^2 \mathbf{R}}{R^2}$$



The meridional and vertical components of these forces are:

$$\left(\frac{Dv}{Dt}\right) = -2\Omega u \sin\phi - \frac{u^2}{a}\tan\phi$$

$$\left(\frac{Dw}{Dt}\right) = 2\Omega u \cos\phi + \frac{u^2}{a}$$

much smaller than gravitational force

Summary of Coriolis Force

$$\left(\frac{Du}{Dt}\right) = \left(2\Omega \sin\phi + \frac{u}{a}\tan\phi\right)\frac{Dy}{Dt} = 2\Omega v \sin\phi + \frac{uv}{a}\tan\phi$$

$$\left(\frac{Dv}{Dt}\right) = -2\Omega u \sin\phi - \frac{u^2}{a}\tan\phi$$

For synoptic-scale motion, $|u| \ll \Omega R$ **, therefore:**

$$\left(\frac{Du}{Dt}\right)_{Co} = 2\Omega v \sin\phi = fv$$

$$\left(\frac{Dv}{Dt}\right)_{Co} = -2\Omega u \sin\phi = -fu$$

$$\longrightarrow \left(\frac{D\mathbf{V}}{Dt}\right)_{Co} = -f\mathbf{k} \times \mathbf{V}$$

where $f = 2\Omega \sin\phi$ is the *Coriolis parameter*.