Lecture 3: Applications of Basic Equations

- Pressure Coordinates: Advantage and Disadvantage
- Momentum Equation \( \Rightarrow \) Balanced Flows
- Thermodynamic & Momentum Eq.s \( \Rightarrow \) Thermal Wind Balance
- Continuity Equation \( \Rightarrow \) Surface Pressure Tendency
- Trajectories and Streamlines
- Ageostrophic Motion

Pressure as Vertical Coordinate

- From the hydrostatic equation, it is clear that a single valued monotonic relationship exists between pressure and height in each vertical column of the atmosphere.
- Thus we may use pressure as the independent vertical coordinate.
- Horizontal partial derivatives must be evaluated holding \( p \) constant.
- How to treat the horizontal pressure gradient force?

Horizontal Derivatives on Pressure Coordinate

\[
\frac{(p_0 + \delta p) - p_0}{\delta x} = \frac{(p_0 + \delta p) - p_0}{\delta z} \left( \frac{\delta z}{\delta x} \right)_p
\]

\[
\frac{\partial p}{\partial x} \bigg|_z = - \frac{\partial p}{\partial z} \bigg|_x \frac{\partial z}{\partial x} \bigg|_p
\]

Using the hydrostatic balance equation

\[
- \frac{1}{\rho} \frac{\partial p}{\partial x} = -g \frac{\partial z}{\partial x} \bigg|_p = - \frac{\partial \Phi}{\partial x} \bigg|_p
\]

- x-component of pressure gradient force = \( F_x = \frac{1}{m} \frac{\partial p}{\partial x} = - \frac{\partial \Phi}{\partial x} \bigg|_p \)
- y-component of pressure gradient force = \( F_y = \frac{1}{m} \frac{\partial p}{\partial y} = - \frac{\partial \Phi}{\partial y} \bigg|_p \)
Horizontal Momentum Eq. Scaled for Midlatitude Synoptic-Scale

Z-Coordinate

\[
\frac{d\mathbf{V}}{dt} = -2\Omega \times \mathbf{V} - \frac{1}{\rho} \nabla P
\]

P-Coordinate

\[
\frac{d\mathbf{V}}{dt} = -2\Omega \times \mathbf{V} - \nabla \Phi
\]

Advantage of Using P-Coordinate

• Thus in the isobaric coordinate system the horizontal pressure gradient force is measured by the gradient of geopotential at constant pressure.

• Density no longer appears explicitly in the pressure gradient force; this is a distinct advantage of the isobaric system.

• Thus, a given geopotential gradient implies the same geostrophic wind at any height, whereas a given horizontal pressure gradient implies different values of the geostrophic wind depending on the density.

Geostrophic Approximation, Balance, Wind

Scaling for mid-latitude synoptic-scale motion

\[
-fv \approx -\frac{1}{\rho} \frac{\partial p}{\partial x}; \quad fu \approx -\frac{1}{\rho} \frac{\partial p}{\partial y}
\]

\[
V_\sigma \equiv k \times \frac{1}{\rho f} \nabla p
\]

Geostrophic wind

• The fact that the horizontal flow is in approximate geostrophic balance is helpful for diagnostic analysis.

Vertical Velocity in P-Coordinate

• Vertical Velocity in the Z-coordinate is \( w \), which is defined as \( \frac{dz}{dt} \):

  \( w > 0 \) for ascending motion
  \( w < 0 \) for descending motion

• Vertical velocity in the P coordinate is \( \omega \) (pronounced as “omega”), which is defined as \( \frac{dp}{dt} \):

  \( \omega < 0 \) for ascending motion
  \( \omega > 0 \) for descending motion
Continuity Eq. on P-Coordinate

• Following a control volume \( \delta V = \delta x \delta y \delta z = -\delta \rho \delta x \delta y / \rho g \) using hydrostatic balance, the mass of the volume does not change:

\[
\frac{1}{\delta M} \frac{d(\delta M)}{dt} = 0 = \frac{-\delta \rho \delta x \delta y / \rho g}{\delta \rho \delta x \delta y / \rho g}.
\]

\[
\frac{1}{\delta V} \frac{d(\delta V)}{dt} = \frac{\delta x \delta y \delta p}{\delta x \delta y \delta p}, \quad \text{and} \quad \frac{\delta (\rho \delta p)}{\delta t} = 0.
\]

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial p} = 0 \quad \text{Simpler! No density variations involved in this form of continuity equation.}
\]

Thermodynamic Eq. on P-Coordinate

\[
\left( \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) - S_p \delta \rho = J / c_p
\]

\[
S_p = \frac{RT}{c_p} \frac{\partial \rho}{\partial p} - \frac{T}{\rho \partial \rho} = -T \frac{\partial \rho}{\partial p} = (\Gamma_d - \Gamma) / \rho g
\]

• This form is similar to that on the Z-coordinate, except that there is a strong height dependence of the stability measure \( S_p \), which is a minor disadvantage of isobaric coordinates.

Velocity Divergence Form (Lagrangian View)

• Following a control volume of a fixed mass \( \delta M \), the amount of mass is conserved.

\[
\frac{1}{\delta V} \frac{d(\delta V)}{dt} = \frac{1}{\rho} \frac{D \delta V}{D t} = \frac{1}{\rho} \frac{D \delta V}{D t} = \frac{1}{\rho} \frac{D \delta V}{D t} = 0
\]

\[
\frac{1}{\delta V} \frac{d(\delta V)}{dt} = \frac{1}{\rho} \frac{D \delta V}{D t} = \frac{1}{\rho} \frac{D \delta V}{D t} = 0
\]

\[
\lim_{\delta V \to \delta V} \left[ \frac{1}{\delta V} \frac{D \delta V}{D t} \right] = \frac{\delta u}{\delta x} + \frac{\delta v}{\delta y} + \frac{\delta w}{\delta z} = \nabla \cdot U
\]

\[
\frac{1}{\rho} \frac{D \delta V}{D t} = \nabla \cdot U = 0
\]

Scaling of the Thermodynamic Eq.

\[
c_p \frac{\partial T}{\partial t} = \frac{\partial \rho}{\partial t} = \frac{T}{\partial t}
\]

\[
\frac{\partial T}{\partial t} = \frac{\partial \rho}{\partial t} = \frac{T}{\partial t}
\]

\[
\Gamma = -\frac{\partial T}{\partial z} = \text{lapse rate}
\]

\[
\Gamma_d = -\frac{g}{c_p} = \text{dry lapse rate}
\]

\[
\frac{\partial T}{\partial z} = \frac{\partial \rho}{\partial t} = \frac{T}{\partial t}
\]

\[
\frac{\partial T}{\partial z} = \frac{\partial \rho}{\partial t} = \frac{T}{\partial t}
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Despite the apparent complexity of atmospheric motion systems as depicted on synoptic weather charts, the pressure (or geopotential height) and velocity distributions in meteorological disturbances are actually related by rather simple approximate force balances.

**Rossby Number**

\[ R_0 = \frac{U}{f_0 L} \]

- Rossby number is a non-dimensional measure of the magnitude of the acceleration compared to the Coriolis force:
- The smaller the \( R(U^2/L)(f_0U) \) the better the geostrophic balance can be used.

**Geostrophic Motion**

“Geo” \( \rightarrow \) Earth
“Strophe” \( \rightarrow \) Turing

At any point on a horizontal surface, we can define a pair of a system of natural coordinates \((t, n)\), where \( t \) is the length directed downstream along the local streamline, and \( n \) is distance directed normal to the streamline and toward the left.
Coriolis and Pressure Gradient Force

- Because the Coriolis force always acts normal to the direction of motion, its natural coordinate form is simply in the following form:
  \[-f \mathbf{k} \times \mathbf{V} = -f \mathbf{V} \mathbf{n}\]

- The pressure gradient force can be expressed as:
  \[-\nabla_p \Phi = -\left( \frac{\partial \Phi}{\partial s} + n \frac{\partial \Phi}{\partial n} \right)\]

Acceleration Term in Natural Coordinate

\[
\frac{DV}{Dt} = \frac{D(Vt)}{Dt} = t \frac{DV}{Dt} + V \frac{Dt}{Dt}
\]

\[
\frac{Dt}{Dt} = \frac{D\mathbf{t}}{Dt} = \frac{DS}{Dt} = \frac{n}{R} V
\]

\[
\delta \psi = \frac{\delta s}{|R|} = \frac{|\delta t|}{|t|} = |\delta t|
\]

\[
\Rightarrow \quad \frac{DV}{Dt} = t \frac{DV}{Dt} + n \frac{V^2}{R}
\]

Therefore, the acceleration following the motion is the sum of the rate of change of speed of the air parcel and its centripetal acceleration due to the curvature of the trajectory.

Horizontal Momentum Eq. (on Natural Coordinate)

\[
\frac{DV}{Dt} = -\frac{\partial \Phi}{\partial s}
\]

\[
\frac{V^2}{R} + fV = -\frac{\partial \Phi}{\partial n}
\]

Geostrophic Balance

\[
fV_g = -\frac{\partial \Phi}{\partial n}
\]

Flow in a straight line (R → ±∞) parallel to height contours is referred to as geostrophic motion. In geostrophic motion the horizontal components of the Coriolis force and pressure gradient force are in exact balance so that \( V = V_g \).
Cyclostrophic Balance

\[ \frac{V^2}{R} = -\frac{\partial \Phi}{\partial n} \Rightarrow V = \left( -\frac{R \partial \Phi}{\partial n} \right)^{1/2} \]

Only low pressure system can have cyclostrophic flow.

- If the horizontal scale of a disturbance is small enough, the Coriolis force may be neglected compared to the pressure gradient force and the centrifugal force. The force balance normal to the direction of flow becomes in cyclostrophic balance.
- An example of cyclostrophic scale motion is tornado.
- A cyclostrophic motion can be either clockwise or counter-clockwise.

Gradient Balance

\[ \frac{V^2}{R} + fV = -\frac{\partial \Phi}{\partial n} \]

- Horizontal frictionless flow that is parallel to the height contours so that the tangential acceleration vanishes (\( DV/DT = 0 \)) is called gradient flow.
- Gradient flow is a three-way balance among the Coriolis force, the centrifugal force, and the horizontal pressure gradient force.
- The gradient wind is often a better approximation to the actual wind than the geostrophic wind.

Super- and Sub-Geostrophic Wind

- For high pressure system
  - gradient wind > geostrophic wind
  - supergeostrophic.
- For low pressure system
  - gradient wind < geostrophic wind
  - subgeostrophic.

Upper Tropospheric Flow Pattern

- Upper tropospheric flows are characterized by trough (low pressure; isobars dip southward) and ridge (high pressure; isobars bulge northward).
- The winds are in gradient wind balance at the bases of the trough and ridge and are slower and faster, respectively, than the geostrophic winds.
- Therefore, convergence and divergence are created at different parts of the flow patterns, which contribute to the development of the low and high systems.
Convergence/Divergence and Vertical Motion

- Convergence in the upper tropospheric flow pattern can cause descending motion in the air column ➔ surface pressure increase (high pressure) ➔ clear sky
- Divergence in the upper tropospheric flow pattern can cause ascending motion in the air column ➔ surface pressure decreases (low pressure) ➔ cloudy weather

Convergence/Divergence in Jetstreak

- The convergence/divergence produced by the curvature and jetstreak effects cancels each other to the south of the jetstream axis but enhances each other to the north of the jetstream.
- The strongest divergence aloft occurs on the northeast side of the trough, where a surface low pressure tends to develop.
- The strongest convergence aloft occurs on the northwest side of the trough, where a surface high pressure tends to develop. However, other processes are more important than this upper-level convergence in affecting the development of high pressure system.

Developments of Low- and High-Pressure Centers

- **Dynamic Effect:** Combined curvature and jetstreak effects produce upper-level convergence on the west side of the trough and favors the formation of a low-level low-pressure center. The same combined effects produce a upper-level divergence on the east side of the trough and favors the formation of a low-level low-pressure center.
- **Thermodynamic Effect:** heating ➔ surface low pressure; cooling ➔ surface high pressure.
- **Frictional Effect:** Surface friction will cause convergence into the low-pressure center after it is produced by upper-level dynamic effects, which adds air mass into the vertical air column and tend to produce a surface high-pressure center. Surface friction again tends to destroy the surface high center.
- **Low Pressure:** The evolution of a low center depends on the relative strengths of the upper-level development and low-level friction damping.
- **High Pressure:** The development of a high center is controlled more by the convergence of surface cooling than by the upper-level dynamic effects. Surface friction again tends to destroy the surface high center.
Trajectory and Streamline

- It is important to distinguish clearly between streamlines, which give a “snapshot” of the velocity field at any instant, and trajectories, which trace the motion of individual fluid parcels over a finite time interval.
- The geopotential height contour on synoptic weather maps are streamlines not trajectories.
- In the gradient balance, the curvature (R) is supposed to be the estimated from the trajectory, but we estimate from the streamlines from the weather maps.

Thermal Wind Balance

(1) Geostrophic Balance
\[ \frac{\partial \Phi}{\partial x} \text{ and } \frac{\partial \Phi}{\partial y} \]

(2) Hydrostatic Balance
\[ \frac{\partial \Phi}{\partial p} = -\rho = \frac{RT}{p} \]

Combine (1) and (2) ➔

\[ \frac{\partial V_g}{\partial p} = \frac{\rho}{\partial \ln p} = -\frac{R}{f} \frac{\partial T}{\partial x} \]

Vertical Motions

- For synoptic-scale motions, the vertical velocity component is typically of the order of a few centimeters per second. Routine meteorological soundings, however, only give the wind speed to an accuracy of about a meter per second.
- Thus, in general the vertical velocity is not measured directly but must be inferred from the fields that are measured directly.
- Two commonly used methods for inferring the vertical motion field are (1) the kinematic method, based on the equation of continuity, and (2) the adiabatic method, based on the thermodynamic energy equation.
The Kinematic Method

- We can integrate the continuity equation in the vertical to get the vertical velocity.

\[
\left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)_z + \frac{\partial \omega}{\partial z} = 0
\]

\[
\omega(p) = \omega(p_0) - \int_{p_0}^p \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dp
\]

- We use the information of horizontal divergence to infer the vertical velocity. However, for midlatitude weather, the horizontal divergence is due primarily to the small departures of the wind from geostrophic balance. A 10% error in evaluating one of the wind components can easily cause the estimated divergence to be in error by 100%.

- For this reason, the continuity equation method is not recommended for estimating the vertical motion field from observed horizontal winds.

The Adiabatic Method

- The adiabatic method is not so sensitive to errors in the measured horizontal velocities, is based on the thermodynamic energy equation.

\[
\left( \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) - S \rho \omega = \frac{J}{c_p}
\]

\[
\omega = S_p^{-1} \left( \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right)
\]

Barotropic and Baroclinic Atmosphere

**Barotropic Atmosphere**
- no temperature gradient on pressure surfaces
- isobaric surfaces are also the isothermal surfaces
- density is only function of pressure \( \rho = \rho(p) \)
- no thermal wind
- no vertical shear for geostrophic winds
- geostrophic winds are independent of height
- you can use a one-layer model to represent the barotropic atmosphere

**Baroclinic Atmosphere**
- temperature gradient exists on pressure surfaces
- density is function of both pressure and temperature \( \rho = \rho(p, T) \)
- thermal wind exists
- geostrophic winds change with height
- you need a multiple-layer model to represent the baroclinic atmosphere