Waves in the Atmosphere and Oceans

Restoring Force

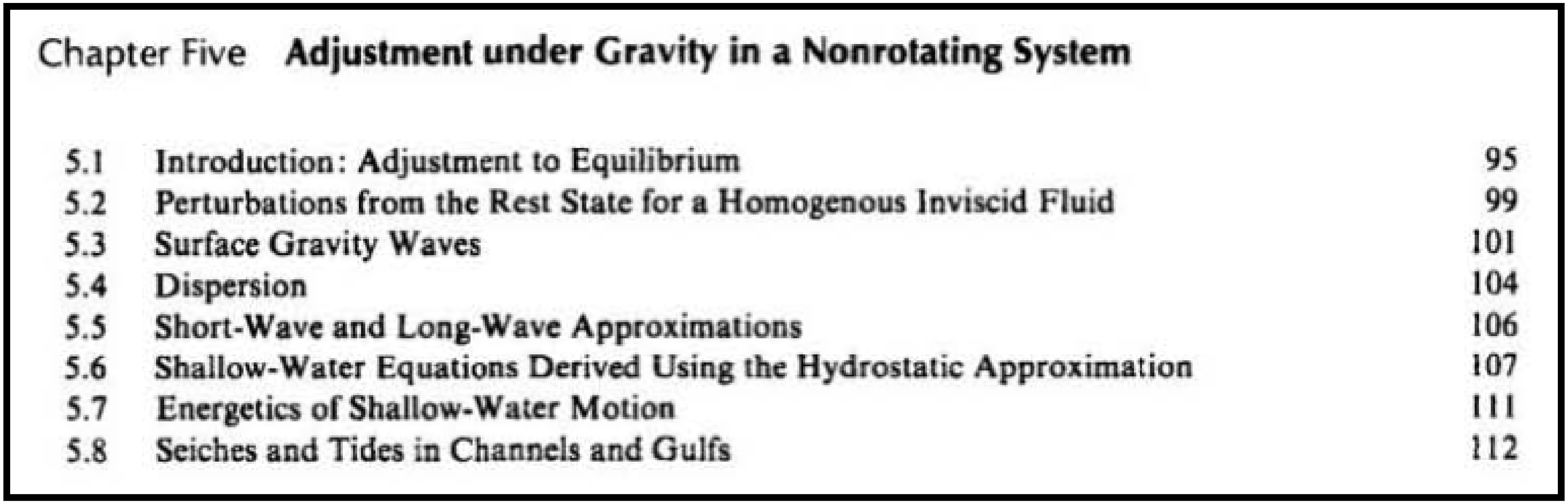
☐ Conservation of potential temperature in the presence of positive static stability ➔ internal gravity waves

☐ Conservation of potential vorticity in the presence of a mean gradient of potential vorticity ➔ Rossby waves

- **External gravity wave** (Shallow-water gravity wave)
- **Internal gravity (buoyancy) wave**
- **Inertial-gravity wave**: Gravity waves that have a large enough wavelength to be affected by the earth's rotation.
- **Rossby Wave**: Wavy motions results from the conservation of potential vorticity.
- **Kelvin wave**: It is a wave in the ocean or atmosphere that balances the Coriolis force against a topographic boundary such as a coastline, or a waveguide such as the equator. Kelvin wave is non-dispersive.

Lecture 6: Adjustment under Gravity in a Non-Rotating System

- Overview of Gravity waves
- Surface Gravity Waves
- “Shallow” Water
- Shallow-Water Model
- Dispersion

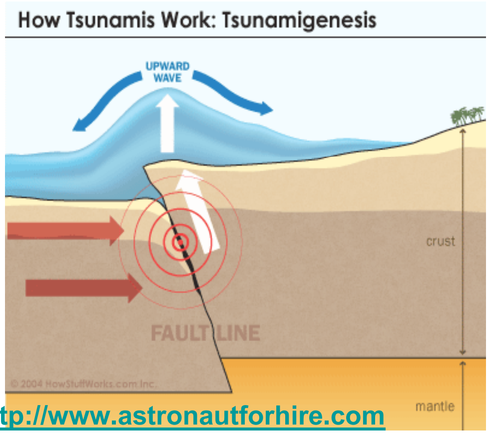


Chapter Five **Adjustment under Gravity in a Nonrotating System**

Goals of this Chapter

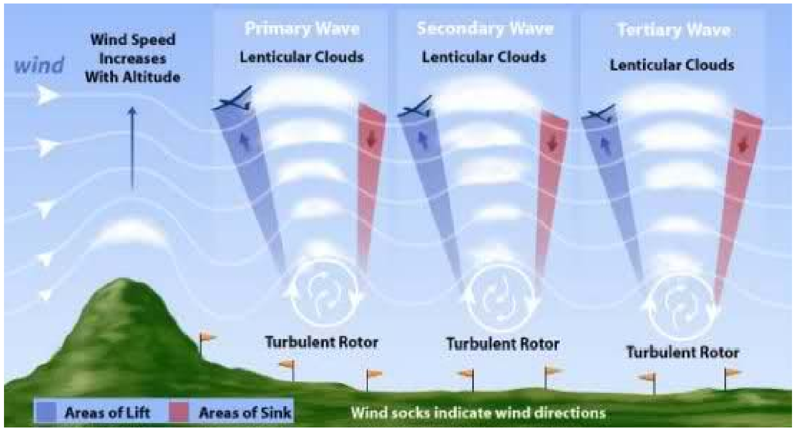
- This chapter marks the beginning of more detailed study of the way the atmosphere-ocean system *tends to adjust to equilibrium.*
- The adjustment processes are most easily understood in the absence of driving forces. Suppose, for instance, that the sun is "switched off," leaving the atmosphere and ocean with some non-equilibrium distribution of properties.
- How will they respond to the gravitational restoring force?
- Presumably there will be an adjustment to some sort of equilibrium. If so, what is the nature of the equilibrium?
- In this chapter, complications due to the rotation and shape of the earth will be ignored and only small departures from the hydrostatic equilibrium will be considered.
- The nature of the adjustment processes will be found by deduction from the equations of motion

Gravity Waves



http://www.astronautforhire.com

http://skywarn256.wordpress.com

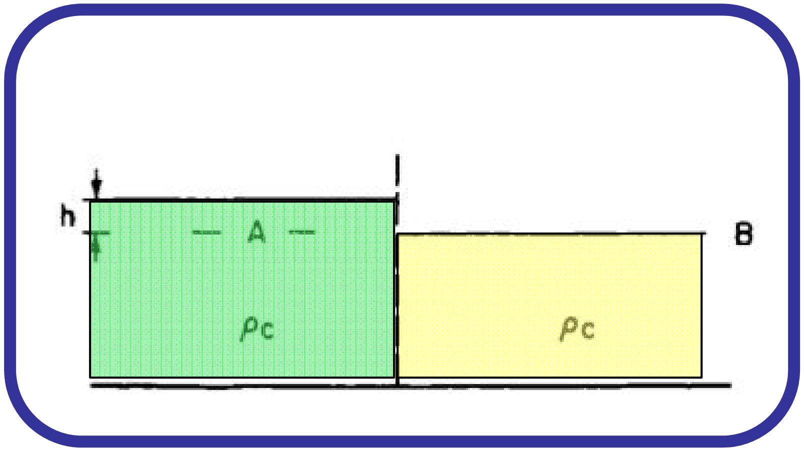


- Gravity waves are waves generated in a fluid medium or at the interface between two media (e.g., the atmosphere and the ocean) which has the restoring force of gravity or buoyancy.
- When a fluid element is displaced on an interface or internally to a region with a different density, gravity tries to restore the parcel toward equilibrium resulting in an oscillation about the equilibrium state or wave orbit.
- Gravity waves on an air-sea interface are called surface gravity waves or surface waves while internal gravity waves are called internal waves.

Adjustment Under Gravity in a Non-Rotating System

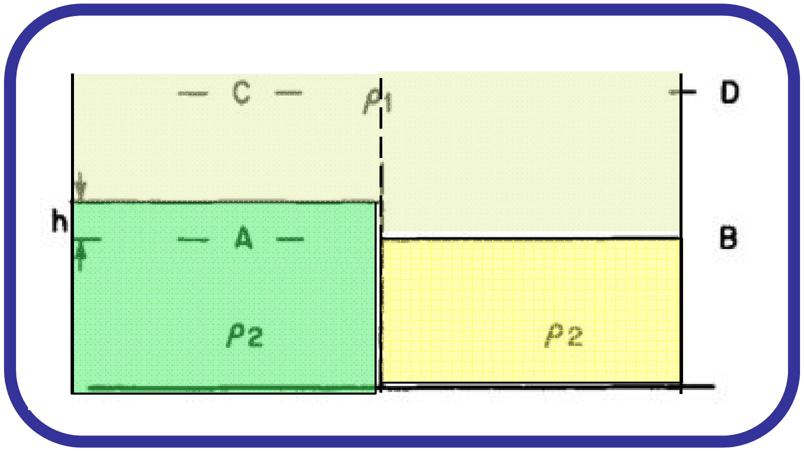
External Gravity Waves

adjustment of a homogeneous fluid with a free surface

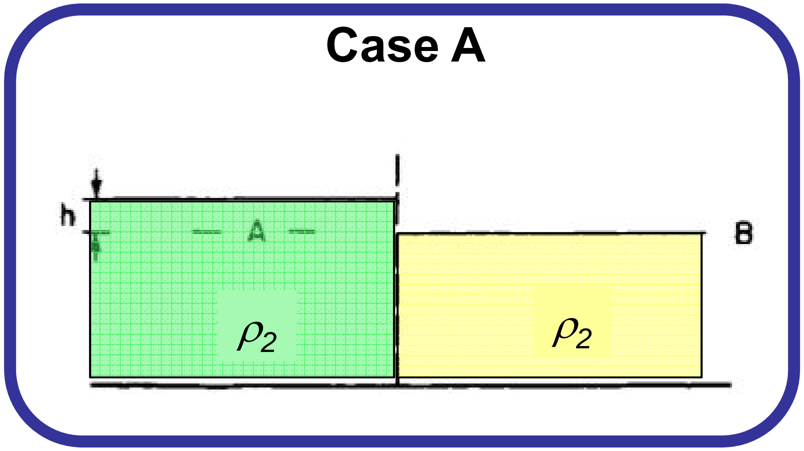


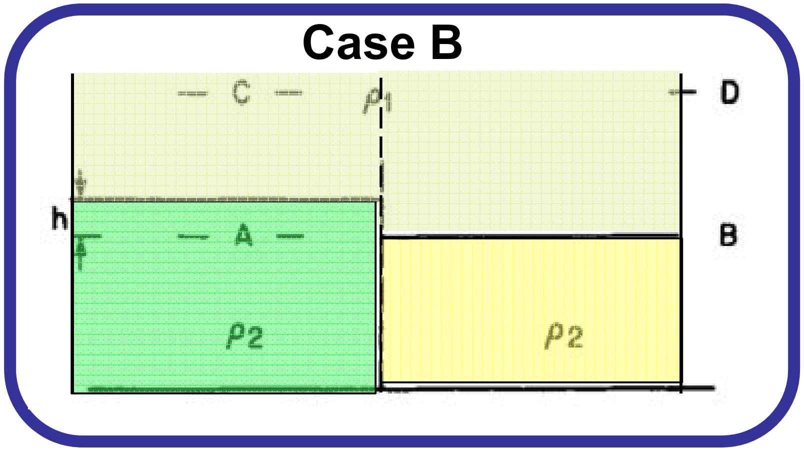
Internal Gravity Waves

adjustment of a density-stratified fluid



Reduced Gravity





Pressure difference between A and B:

$$\Delta P = \rho_2 * g * h$$

Pressure difference between A and B:

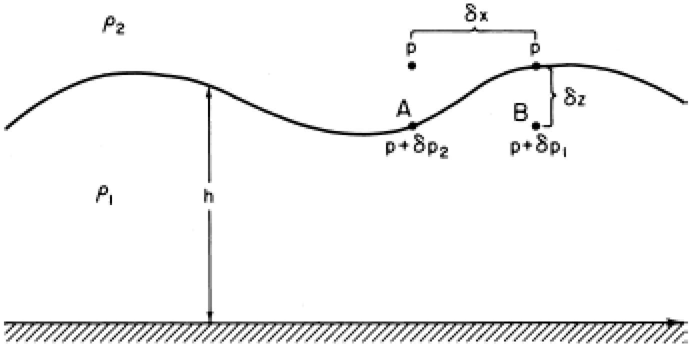
$$\Delta P = \underline{(\rho_2 - \rho_1)} * g * h$$

buoyancy force =
density difference * g

The adjustment process in Case B is exactly the same as in the Case A, except the gravitational acceleration is reduced to a value *g'*, where

$$\boxed{g' = g(\rho_2 - \rho_1)/\rho_2.}$$

A Two-Layer Fluid System



We assume that the motion is two dimensional in the x, z plane.

$\delta\rho = \rho_1 - \rho_2.$

Momentum Equation

P$_A$= $p+\delta p_1 = p+\rho_1 g\delta z = p+\rho_1 g\left(\partial h/\partial x\right)\delta x$

P$_B$= $p+\delta p_2 = p+\rho_2 g\delta z = p+\rho_2 g\left(\partial h/\partial x\right)\delta x$

- ❑ PG $= \lim_{\delta x\to 0}\left[\frac{(p+\delta p_1)-(p+\delta p_2)}{\delta x}\right] = g\delta\rho\frac{\partial h}{\partial x}$
- ❑ Momentum equation:

$$\boxed{\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z} = -\frac{g\delta\rho}{\rho_1}\frac{\partial h}{\partial x}}$$

Continuity Equation

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0$$

➔ $w(h)-w(0) = -h\left(\partial u/\partial x\right)$ $\begin{cases} w(h) = \frac{Dh}{Dt} = \frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x} \\ w(0)=0 \end{cases}$

➔ $\boxed{\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x}+h\frac{\partial u}{\partial x} = \frac{\partial h}{\partial t}+\frac{\partial}{\partial x}(hu)=0}$

$$\boxed{u=\overline{u}+u', \quad h=H+h'}$$

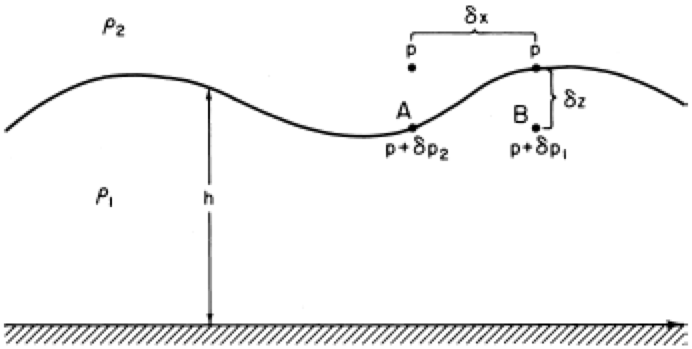
applying the perturbation method

$$\frac{\partial u'}{\partial t}+\overline{u}\frac{\partial u'}{\partial x}+\frac{g\delta\rho}{\rho_1}\frac{\partial h'}{\partial x}=0$$

$$\frac{\partial h'}{\partial t}+\overline{u}\frac{\partial h'}{\partial x}+H\frac{\partial u'}{\partial x}=0$$

➔ $$\boxed{\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)^2 h' - \frac{gH\delta\rho}{\rho_1}\frac{\partial^2 h'}{\partial x^2}=0}$$

Shallow Water Gravity Wave



We assume that the motion is two dimensional in the x, z plane.

$$\delta\rho = \rho_1 - \rho_2.$$

Governing Equation

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)^2 h' - \frac{gH\delta\rho}{\rho_1}\frac{\partial^2 h'}{\partial x^2} = 0$$

$$h' = A\exp[ik(x - ct)]$$

$$c = \overline{u} \pm \left(gH\delta\rho/\rho_1\right)^{1/2}$$

$\delta\rho \approx \rho_1$ (e.g., air and water)

$$c = \overline{u} \pm \sqrt{gH}$$

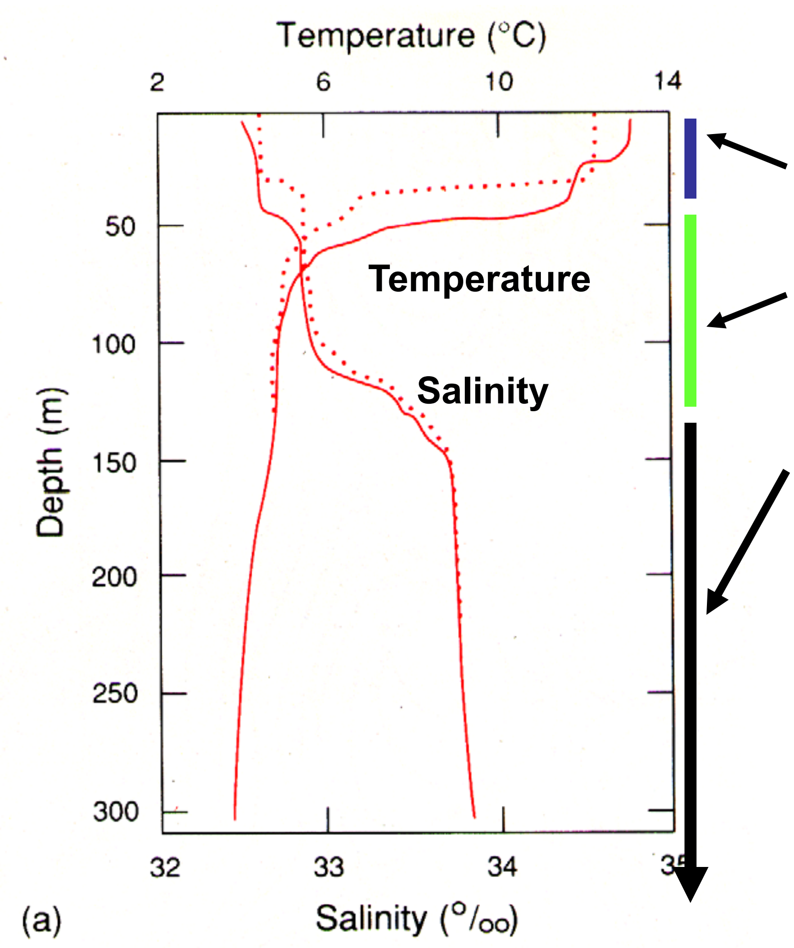
- The shallower the water, the slower the wave.
- Shallow water gravity waves are non-dispersive.

Shallow water wave speed ≈ 200 ms^{-1} for an ocean depth of 4km

- Shallow water gravity waves may also occur at thermocline where the surface water is separated from the deep water. *(These waves can also referred to as the internal gravity waves).*

- If the density changes by an amount $\delta\rho/\rho 1 \approx 0.01$, across the thermocline, then the wave speed for waves traveling along the thermocline will be only one-tenth of the surface wave speed for a fluid of the same depth.

Vertical Structure of Ocean



Mixed Layer: T and S well mixed by winds

Thermocline: large gradient of T and S

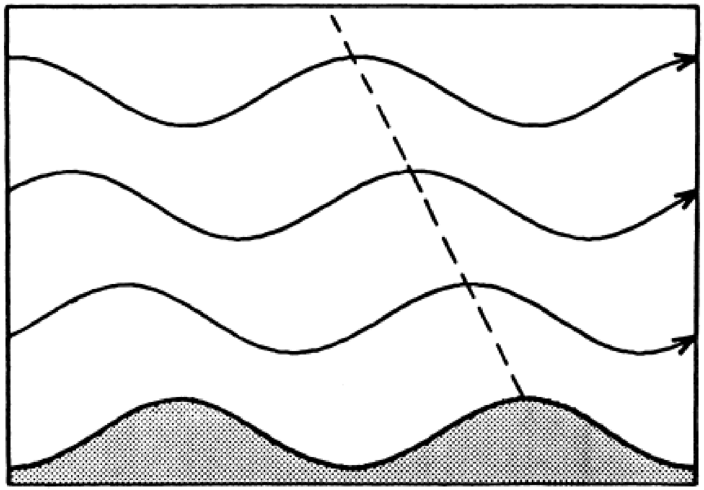
Deep Ocean: T and S independent of height
cold
salty
high nutrient level

(from *Climate System Modeling*)

Shallow and Deep Water

- “Shallow” in this lecture means that the depth of the fluid layer is small compared with the horizontal scale of the perturbation, i.e., the horizontal scale is large compared with the vertical scale.
- Shallow water gravity waves are the ‘long wave approximation” end of gravity waves.
- Deep water gravity waves are the “short wave approximation” end of gravity waves.
- Deep water gravity waves are not important to large-scale motions in the oceans.

Internal Gravity (Buoyancy) Waves



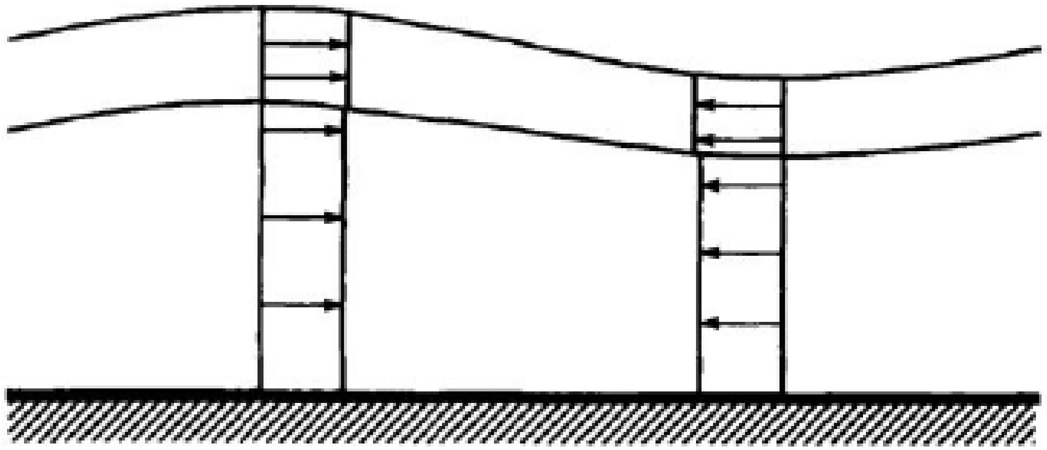
- In a fluid, such as the ocean, which is bounded both above and below, gravity waves propagate primarily in the horizontal plane since vertically traveling waves are reflected from the boundaries to form standing waves.
- In a fluid that has no upper boundary, such as the atmosphere, gravity waves may propagate vertically as well as horizontally. In vertically propagating waves the phase is a function of height. Such waves are referred to as *internal waves.*
- *Although internal* gravity waves are not generally of great importance for synoptic-scale weather forecasting (and indeed are nonexistent in the filtered quasi-geostrophic models), they can be important in mesoscale motions.
- For example, they are responsible for the occurrence of mountain *lee waves. They also are believed to be an important* mechanism for transporting energy and momentum into the middle atmosphere, and are often associated with the formation of clear air turbulence (CAT).

Quasi-Geostrophic Approximation

$$\boxed{\frac{\partial \zeta_g}{\partial t} + \vec{V}_g \bullet \nabla \zeta_g + \beta v_g = -f \nabla \bullet \vec{V}}$$

- Quasi-geostrophic approximation use the geostrophic wind for the actual wind everywhere *except* when computing divergence.
- The Q-G approximation eliminates both sound and gravity waves as solutions to the equations of motion.

Lecture 7: Adjustment under Gravity of a Density-Stratified Fluid

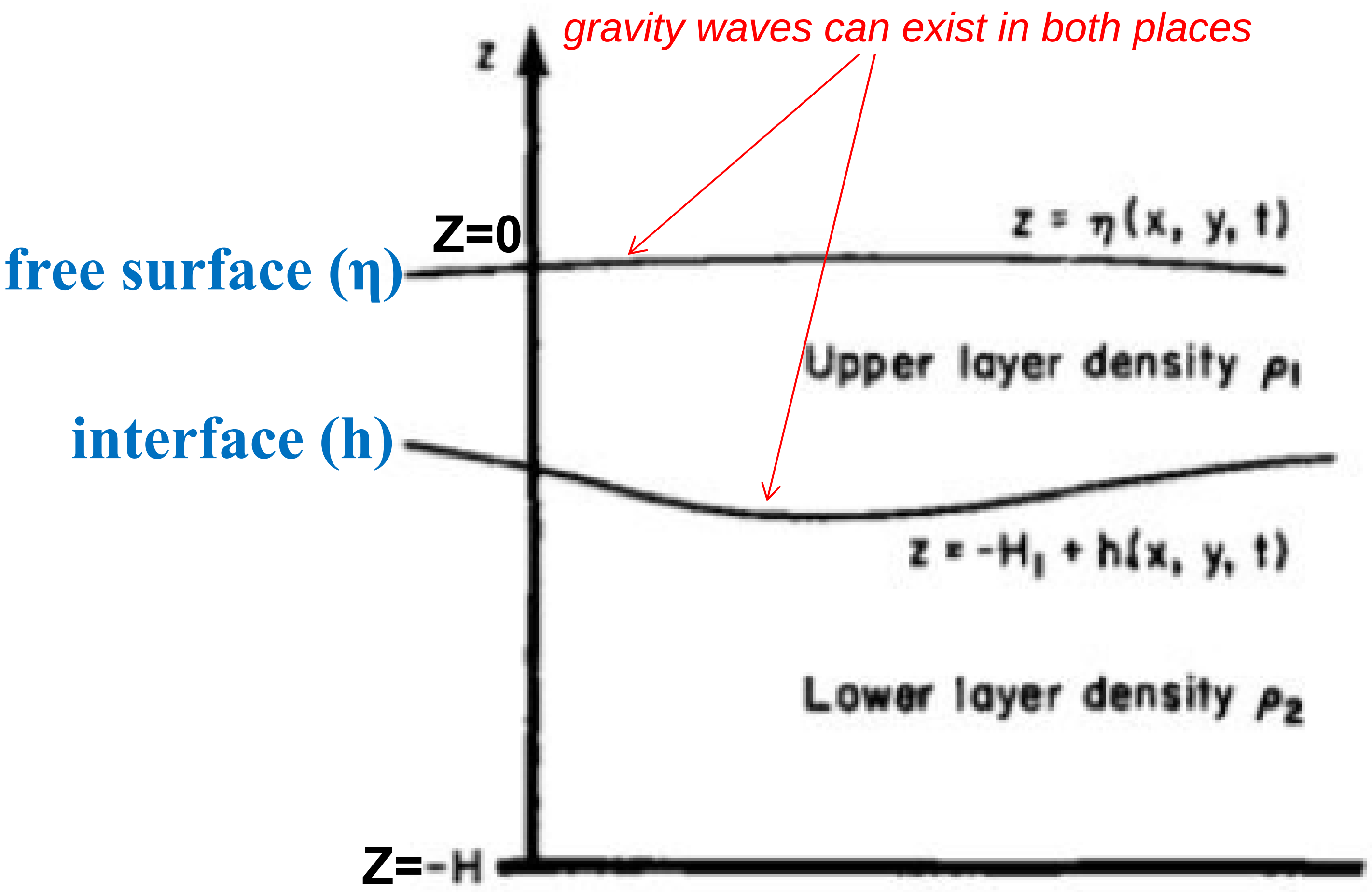


- Normal Mode & Equivalent Depth
- Rigid Lid Approximation
- Boussinesq Approximation
- Buoyancy (Brunt-Väisälä) Frequency
- Dispersion of internal gravity waves

Main Purpose of This Lecture

- As an introduction to the effects of stratification, the case of two superposed shallow layers, each of uniform density, is considered.
- In reality, both the atmosphere and ocean are continuously stratified.
- This serves to introduce the concepts of ***barotropic*** and ***baroclinic*** modes.
- This also serves to introduce two widely used approximations: the ***rigid lid approximation*** and the ***Boussinesq approximation***.

Two Fluids of Different Density



gravity waves can exist in both places

free surface (η)

interface (h)

Z=0

Z=-H

Fluid 1

density = ρ_1

mean depth = H_1

velocity = U_1, V_1

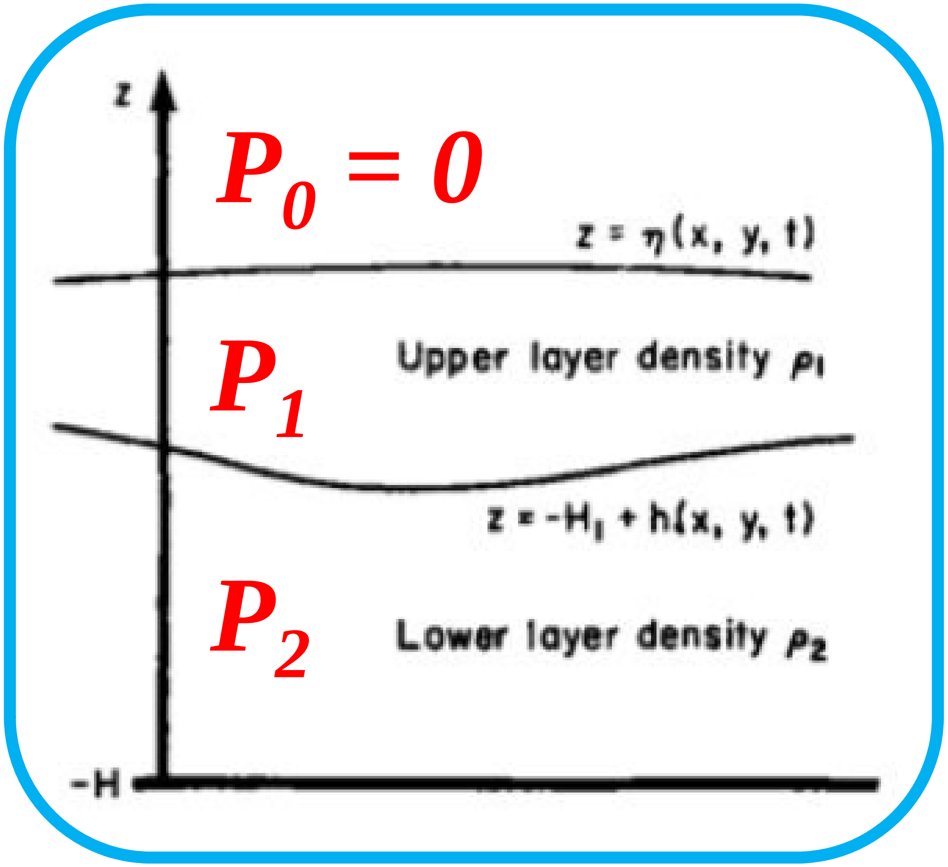
Fluid 2

density = ρ_2

mean depth = H_2

velocity = U_2, V_2

Two Fluids: *Layer 1* $(-H_1 + h < z < \eta)$



$$p_1 = \rho_1 g(\eta - z)$$

❑ **Momentum Equations**

$$\partial u_1/\partial t = -g\,\partial\eta/\partial x,$$

$$\partial v_1/\partial t = -g\,\partial\eta/\partial y,$$

❑**Continuity Equation**

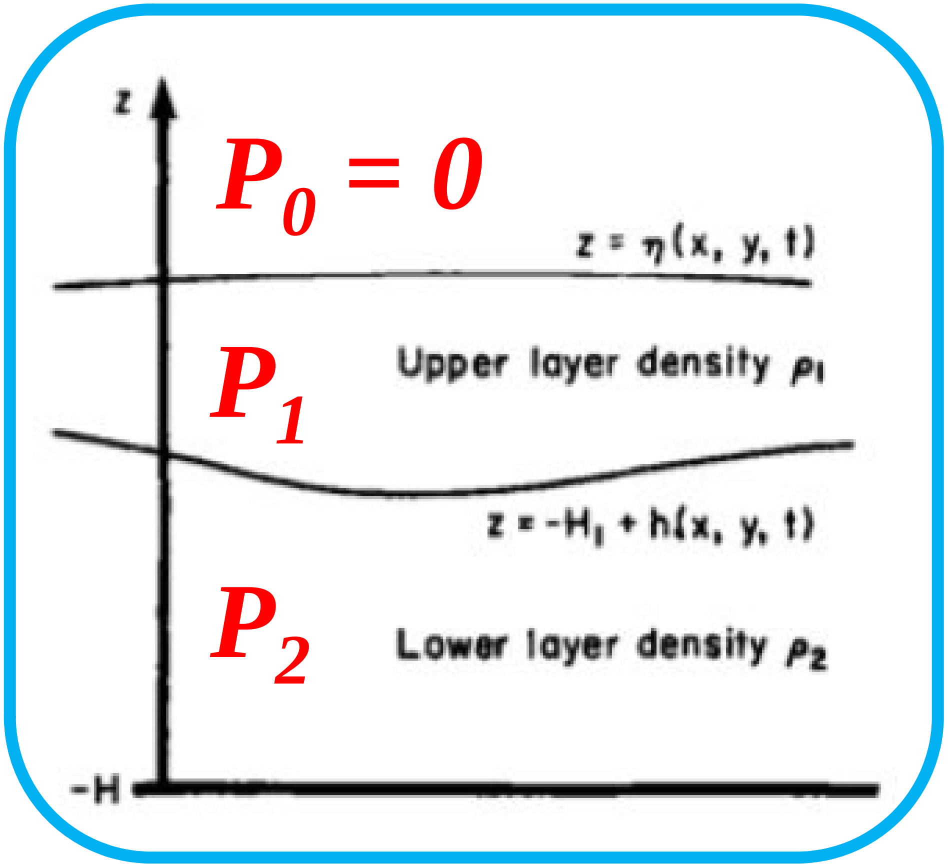
$$\partial(\eta + H_1 - h)/\partial t + H_1(\partial u_1/\partial x + \partial v_1/\partial y) = 0.$$

❑**Taking time derivative of the continuity equation:**

$$\boxed{\frac{\partial^2}{\partial t^2}(\eta - h) = H_1\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)g\eta \equiv gH_1\,\nabla^2\eta}$$

Two Fluids: *Layer 2* ($z < -H_1 + h$)

$$p_2 = \rho_1 g(\eta + H_1 - h) + \rho_2 g(-H_1 + h - z),$$



❑ **Momentum Equations**

$$\frac{\partial u_2}{\partial t} = -\frac{\rho_1}{\rho_2} g \frac{\partial \eta}{\partial x} - g' \frac{\partial h}{\partial x}$$

$$\frac{\partial v_2}{\partial t} = -\frac{\rho_1}{\rho_2} g \frac{\partial \eta}{\partial y} - g' \frac{\partial h}{\partial y}$$

$$g' = g(\rho_2 - \rho_1)/\rho_2$$

= reduced gravity

❑ **Continuity Equation**

$$\partial h/\partial t + H_2(\partial u_2/\partial x + \partial v_2/\partial y) = 0.$$

❑ **Taking time derivative of the continuity**

$$\boxed{\frac{\partial^2 h}{\partial t^2} = H_2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\rho_1}{\rho_2} g\eta + g'h\right) = H_2 \nabla^2(g\eta - g'\eta + g'h),}$$

Adjustments of the Two-Fluid System

- The adjustments in the two-layer fluid system are governed by:

$$\frac{\partial^2}{\partial t^2}(\eta - h) = H_1\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)g\eta \equiv gH_1\,\nabla^2\eta$$

$$\frac{\partial^2 h}{\partial t^2} = H_2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\rho_1}{\rho_2}g\eta + g'h\right) = H_2\,\nabla^2(g\eta - g'\eta + g'h),$$

- Combined these two equations will result in a fourth-order equation, which is difficult to solve.

- This problem can be greatly simplified by looking for solutions which η and h are proportional:

$$\boxed{h(x, y, t) = \mu\eta(x, y, t),}$$

- The governing equations will both reduced to this form:

$$\boxed{\partial^2\eta/\partial t^2 = c_e^2\,\nabla^2\eta,}$$

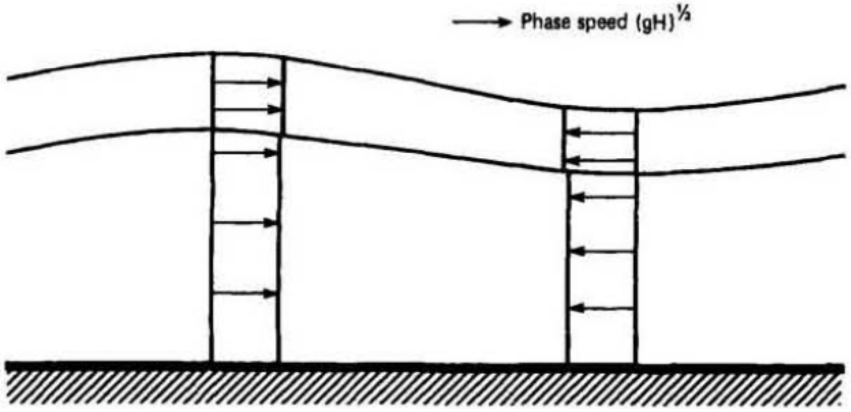
provided that

$$\boxed{gH_1/(1-\mu) = \mu^{-1}(g - g'(1-\mu))H_2 = c_e^2.}$$

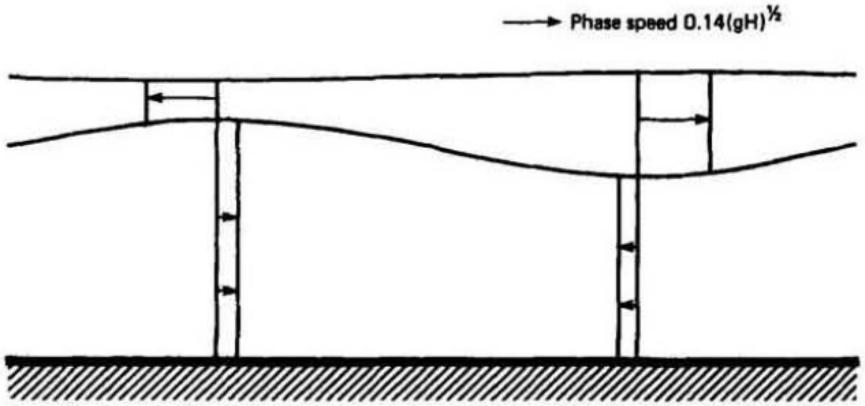
There are two values of μ (and hence two values of c_e) that satisfy this equation.

➔ The motions corresponding to these particular vales are called normal modes of oscillation.

Normal Modes



barotropic mode



baroclinic mode

- The motions corresponding to these particular values of c_e or μ are called ***normal modes*** of oscillation.
- In a system consisting of n layers of different density, there are n normal modes corresponding to the n degrees of freedom.
- A continuously stratified fluid corresponding to an infinite number of layers, and so there is an infinite set of modes.

Structures of the Normal Modes

- The structures of the normal modes can be obtained by solving this equation (from previous slide):

$$c_e^4 - gHc_e^2 + gg'H_1H_2 = 0,$$

$$H = H_1 + H_2$$

- Or, we can use the concept of the one-layer shallow water model, where the phase speed (c) of the gravity wave is related to the depth of the shallow water (H):

$$c = \sqrt{gH}$$

- Using this concept, we can assume each of the normal mode behaves like the one-layer shallow water with a “equivalent depth” of H_e:

$$c_e^2 = gH_e.$$

$$gH_e^2 - gHH_e + g'H_1H_2 = 0.$$

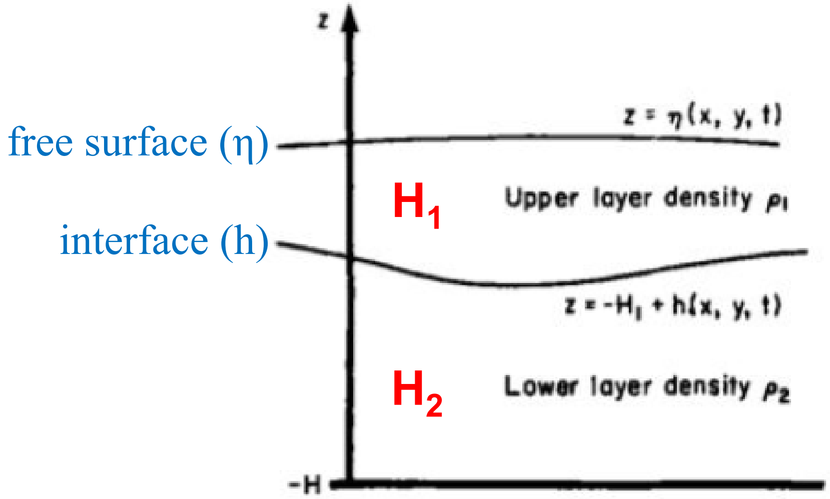
$$c_0^2 = gH(1 - g'H_1H_2/gH^2 \cdots),$$
$$\eta/h \approx H/H_2, \qquad u_2/u_1 = 1 - g'H_1/gH \cdots$$

Solution 1 (brotropic mode)

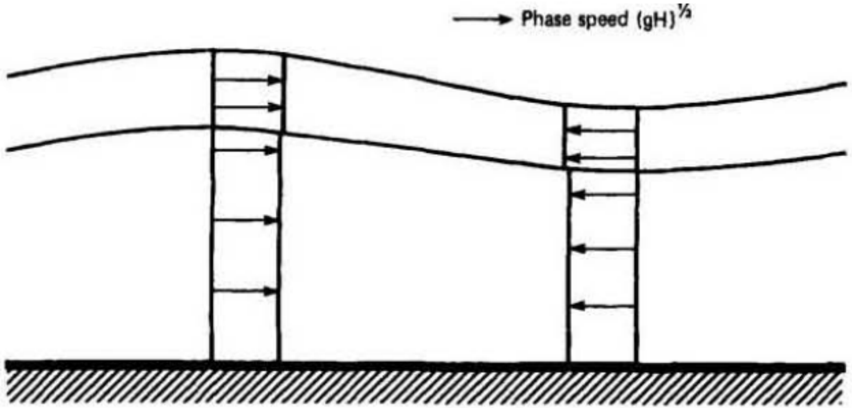
$$c_1^2 = (g'H_1H_2/H)(1 + g'H_1H_2/gH^2 \cdots),$$
$$\eta/h \approx -g'H_2/gH, \qquad u_2/u_1 \approx -H_1/H_2.$$

Solution 2 (baroclinic mode)

Structures of the Normal Modes



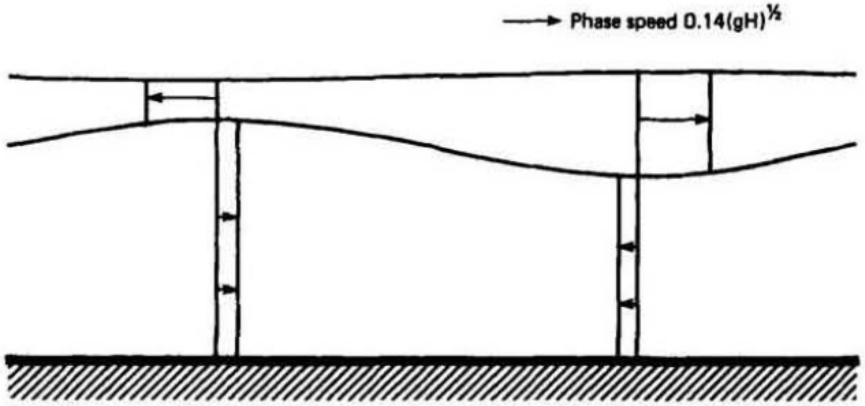
$H=H_1+H_2$



Barotropic mode

$$c_0^2 = gH(1 - g'H_1H_2/gH^2 \cdots),$$
$$\eta/h \approx H/H_2, \qquad u_2/u_1 = 1 - g'H_1/gH \cdots.$$

$\eta > h$; u_2&u_1 of same signs



Baroclinic mode

$$c_1^2 = (g'H_1H_2/H)(1 + g'H_1H_2/gH^2 \cdots),$$
$$\eta/h \approx -g'H_2/gH, \qquad u_2/u_1 \approx -H_1/H_2.$$

$\eta < h$; u_2&u_1 of opposite signs

Equivalent Depth (H_e)

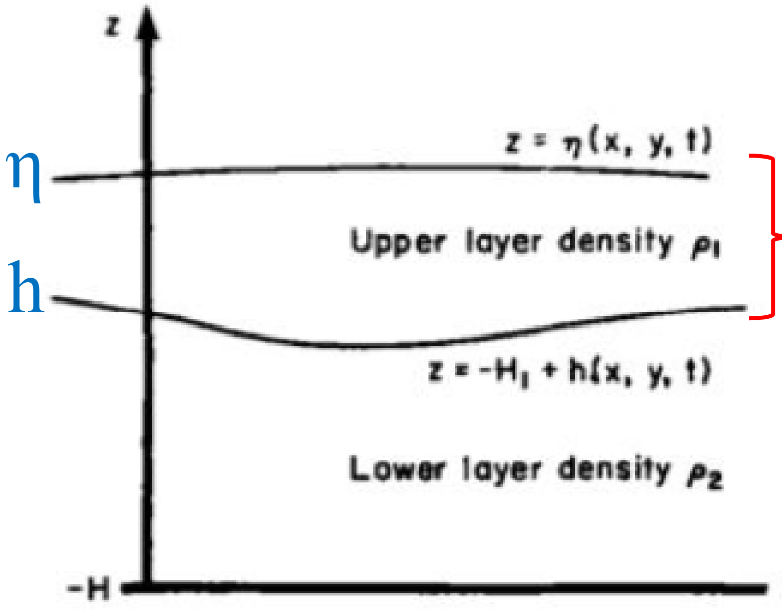
- An N-layer fluid will have one barotropic mode and (N-1) baroclinic modes of gravity waves, each of which has its own equivalent depth.
- Once the equivalent depth is known, we know the dispersion relation of that mode of gravity wave and we know how fast/slow that gravity wave propagates.

$$\boxed{c_e^2 = gH_e.}$$

- For a continuously stratified fluid, it has an infinite number of modes, but not all the modes are imporptant. We only need to identify the major baroclinic modes and to find out their equivalent depths.

Rigid Lid Approximation

(for the upper layer)



$$p_1 = \rho_1 g(\eta - z)$$

☐ **Momentum Equations**

$$\partial u_1/\partial t = -g\ \partial \eta/\partial x,$$

$$\partial v_1/\partial t = -g\ \partial \eta/\partial y,$$

☐ **Continuity Equation**

$$\partial(\eta + H_1 - h)/\partial t + H_1(\partial u_1/\partial x + \partial v_1/\partial y) = 0.$$

- For baroclinic modes, surface displacements (η) are small compared to interface displacements (h).
- If there is a rigid lid at z=0, the identical pressure gradients would have be achieved.

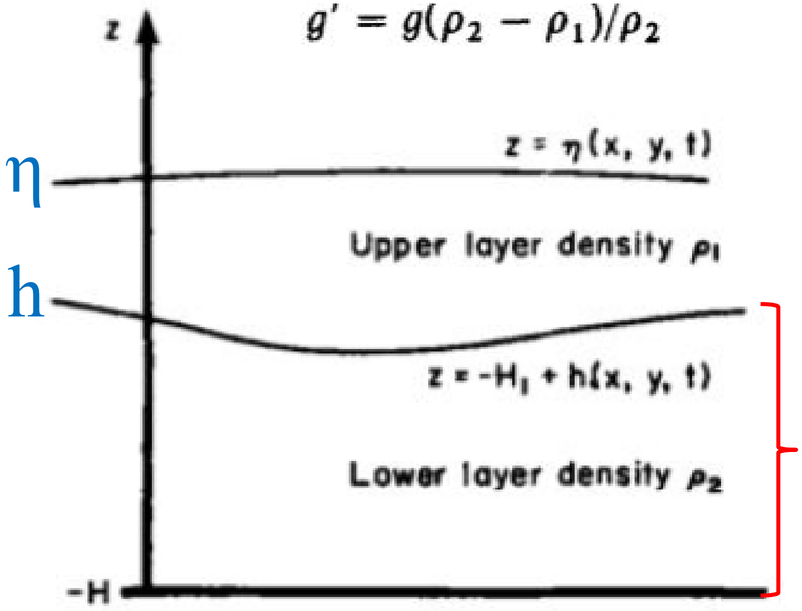
$$\boxed{-\partial h/\partial t + H_1(\partial u_1/\partial x + \partial v_1/\partial y) = 0.}$$

Purpose of Rigid Lid Approximation

- Rigid lid approximation: the upper surface was held fixed but could support pressure changes related to waves of lower speed and currents of interest.
- Ocean models us***ed*** the "rigid lid" approximation to eliminate high-speed external gravity waves and allow a longer time step.
- As a result, ocean tides and other waves having the speed of tsunamis were filtered out.
- The **rigid lid approximation** was used in the 70's to filter out gravity wave dynamics in ocean models. Since then, ocean model have evolved to include a free-surface allowing fast-moving gravity wave physics.

Boussinesq Approximation

(for the lower layer)



❑ **Momentum Equations**

$$\frac{\partial u_2}{\partial t} = -\frac{\rho_1}{\rho_2} g \frac{\partial \eta}{\partial x} - g' \frac{\partial h}{\partial x}$$

$$\frac{\partial v_2}{\partial t} = -\frac{\rho_1}{\rho_2} g \frac{\partial \eta}{\partial y} - g' \frac{\partial h}{\partial y}$$

❑ **Continuity Equation**

$$\partial h/\partial t + H_2(\partial u_2/\partial x + \partial v_2/\partial y) = 0.$$

- Boussinesq approx: replace the ratio (ρ_1/ρ_2) by unity in the momentum equation.

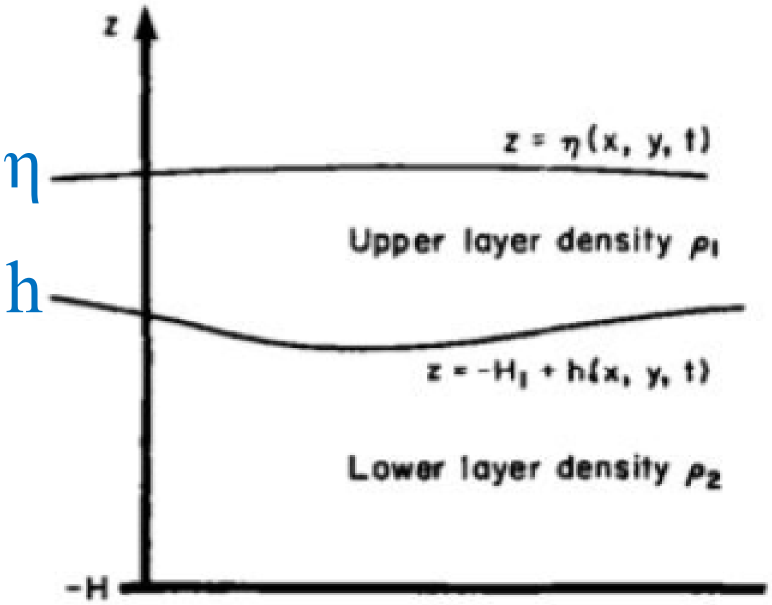
$$\boxed{\begin{aligned} \partial u_2/\partial t &= -g\, \partial \eta/\partial x - g'\, \partial h/\partial x, \\ \partial v_2/\partial t &= -g\, \partial \eta/\partial y - g'\, \partial h/\partial y. \end{aligned}}$$

- We keep the density difference in this g′ term, because it involves density difference (ρ_1- ρ_2)/ ρ_1*g, which is related to the buoyancy force.

Purpose of Boussinesq Approximation

- This approximation states that density differences are sufficiently small to be neglected, except where they appear in terms multiplied by *g*, the acceleration due to gravity (i.e., buoyancy).
- In the Boussinesq approximation, which is appropriate for an almost-incompressible fluid, it assumed that variations of density are small, so that in the intertial terms, and in the continuity equation, we may substitute ρ by ρ_0, a constant. However, even weak density variations are important in buoyancy, and so we retain variations in ρ in the buoyancy term in the vertical equation of motion.
- *Sound waves are impossible/neglected when the Boussinesq approximation is used, because sound waves move via density variations.*
- Boussinesq approximation is for the problems that the variations of temperature as well as the variations of density are small. In these cases, the variations in volume expansion due to temperature gradients will also small. For these case, Boussinesq approximation can simplify the problems and save computational time.

After Using the Two Approximations



☐ **Upper layer**

$$\partial u_1/\partial t = -g\,\partial\eta/\partial x,$$
$$\partial v_1/\partial t = -g\,\partial\eta/\partial y,$$
$$-\partial h/\partial t + H_1(\partial u_1/\partial x + \partial v_1/\partial y) = 0.$$

☐ **Lower layer**

$$\partial u_2/\partial t = -g\,\partial\eta/\partial x - g'\,\partial h/\partial x,$$
$$\partial v_2/\partial t = -g\,\partial\eta/\partial y - g'\,\partial h/\partial y.$$
$$\partial h/\partial t + H_2(\partial u_2/\partial x + \partial v_2/\partial y) = 0.$$

- After the approximations, there is no η in the two continuity equation ➔ They can be combined to become one equation.
- The two momentum equations can also be combined into one single equation without η.
- At the end, the continuity and momentum equations for the upper and lower layers can be combined to solve for the dispersive relation for the baroclinic mode of the gravity wave.

Brunt–Väisälä Frequency (*N*)

❑ Consider a parcel of (water or gas) that has density of ρ_0 and the environment with a density that is a function of height: $\rho = \rho(z)$. If the parcel is displaced by a small vertical increment z', it will subject to an extra gravitational force against its surroundings of:

$$\rho_0 \frac{\partial^2 z'}{\partial t^2} = -g(\rho_0 - \rho(z'))$$

$$\rho(z) - \rho_0 = \frac{\partial \rho(z)}{\partial z} z',$$

$$\rightarrow \frac{\partial^2 z'}{\partial t^2} = \frac{g}{\rho_0} \frac{\partial \rho(z)}{\partial z} z'$$

$$\rightarrow \boxed{z' = z'_0 e^{\sqrt{-N^2}t}}$$

where $\boxed{N = \sqrt{-\frac{g}{\rho_0} \frac{\partial \rho(z)}{\partial z}}}$ $\boxed{N = \sqrt{\frac{g}{\theta} \frac{d\theta}{dz}},}$

for oceans *for the atmosphere*

- A fluid parcel in the presence of stable stratification ($N^2 > 0$) will oscillate vertically if perturbed vertically from its starting position.
- In atmospheric dynamics, oceanography, and geophysics, the Brunt-Vaisala frequency, or buoyancy frequency, is the angular frequency at which a vertically displaced parcel will oscillate within statically stable environment.
- The Brunt–Väisälä frequency relates to internal gravity waves and provides a useful description of atmospheric and oceanic stability.

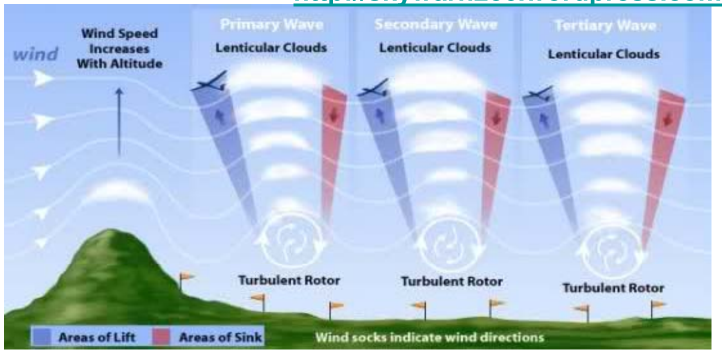
Internal Gravity Waves in Atmosphere and Oceans

In Oceans

$$\boxed{N = \sqrt{-\frac{g}{\rho_0}\frac{\partial \rho(z)}{\partial z}}}$$

In Atmosphere

$$\boxed{N \equiv \sqrt{\frac{g}{\theta}\frac{d\theta}{dz}},}$$



http://skywarn256.wordpress.com

- Internal gravity waves can be found in both the ***statically stable*** (dΘ/dz>0) atmosphere and the ***stably stratified*** (-dρ/dz>0)ocean.
- The buoyancy frequency for the internal gravity wave in the ocean is determined by the vertical density gradient, while it is determined by the vertical gradient of potential temperature.
- In the troposphere, the typical value of N is 0.01 sec^{-1}, which correspond to a period of about 10 minutes.
- Although there are plenty of gravity waves in the atmosphere, most of them have small amplitudes in the troposphere and are not important, except that the gravity waves generated by flows over mountains. These mountain waves can have large amplitudes.
- Gravity waves become more important when they propagate into the upper atmosphere (particularly in the mesosphere) where their amplitudes got amplified due to low air density there.

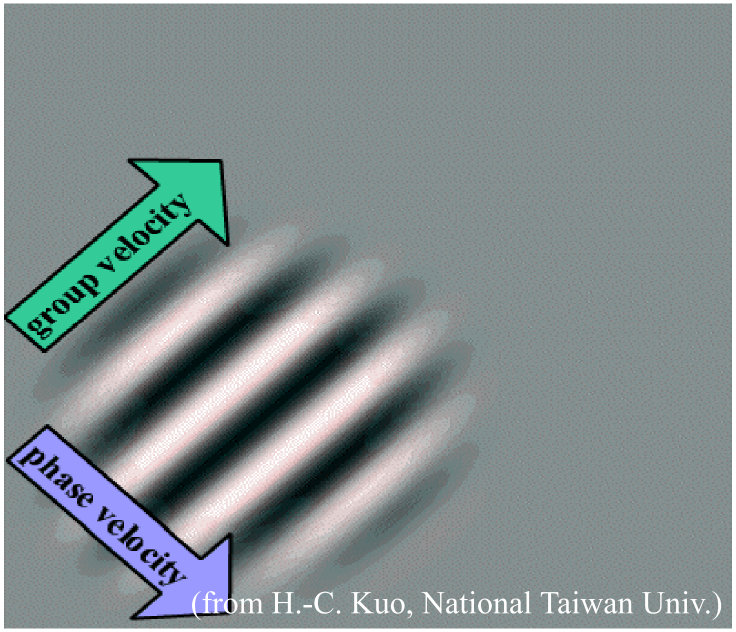
Dispersion of Internal Gravity Waves

mean flow   zonal wavenumber   vertical wavenumber   total wavenumber

$$\hat{\nu} \equiv \nu - \bar{u}k = \pm Nk/\left(k^2 + m^2\right)^{1/2} = \pm Nk/\left|\boldsymbol{\kappa}\right|$$

$\hat{\nu}$ is always smaller than N!!

Internal gravity waves can have any frequency between zero and a maximum value of N.



(from H.-C. Kuo, National Taiwan Univ.)

- In the atmosphere, internal gravity waves generated in the troposphere by cumulus convection, by flow over topography, and by other processes may propagate upward many scale heights into the middle atmosphere.

- Phase velocity:

$$c_x = \hat{\nu}/k \text{ and , } c_z = \hat{\nu}/m$$

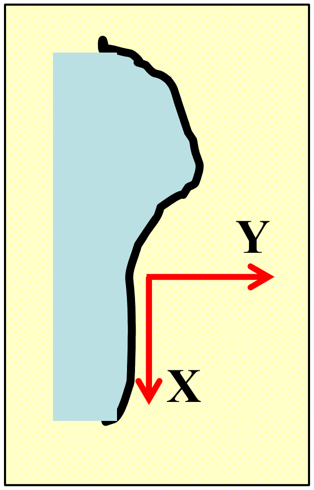
- Group velocity:

$$c_{gx} = \frac{\partial \nu}{\partial k} = \bar{u} \pm \frac{Nm^2}{\left(k^2 + m^2\right)^{3/2}}$$

$$c_{gz} = \frac{\partial \nu}{\partial m} = \pm \frac{(-Nkm)}{\left(k^2 + m^2\right)^{3/2}}$$

- Internal gravity waves thus have the remarkable property that group velocity is perpendicular to the direction of phase propagation.

Kelvin Waves



Governing Equations

$$\begin{aligned} \frac{du}{dt} - fv &= -g\frac{\partial h}{\partial x} \; ; \\ \frac{dv}{dt} + fu &= -g\frac{dh}{dy} \; ; \\ \frac{dh}{dt} + D\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) &= 0 \; . \end{aligned}$$

A unique boundary condition

$\boxed{y = 0 \text{ is } v = 0}$

$$\begin{pmatrix} u' \\ h' \end{pmatrix} = \mathrm{Re}\left\{\begin{pmatrix} U(y) \\ H(y) \end{pmatrix} \exp\left[ik(x - ct)\right]\right\}$$

$$H = const \times \exp\left(-\frac{f}{c}y\right)$$

$$-\frac{f}{g}U = -\frac{f}{c}H$$

$$c = \sqrt{gD}$$

- A Kelvin wave is a type of low-frequency gravity wave in the ocean or atmosphere that balances the Earth's Coriolis force against a topographic boundary such as a coastline, or a waveguide such as the equator.
- Therefore, there are two types of Kelvin waves: coastal and equatorial.
- A feature of a Kelvin wave is that it is non-dispersive, i.e., the phase speed of the wave crests is equal to the group speed of the wave energy for all frequencies.

Costal Kelvin Waves

$$H = const \times \exp\left(-\frac{f}{c}y\right)$$

At the coast $y = 0$ is $v = 0$:

$$c = \sqrt{gD}$$

depth of the fluid

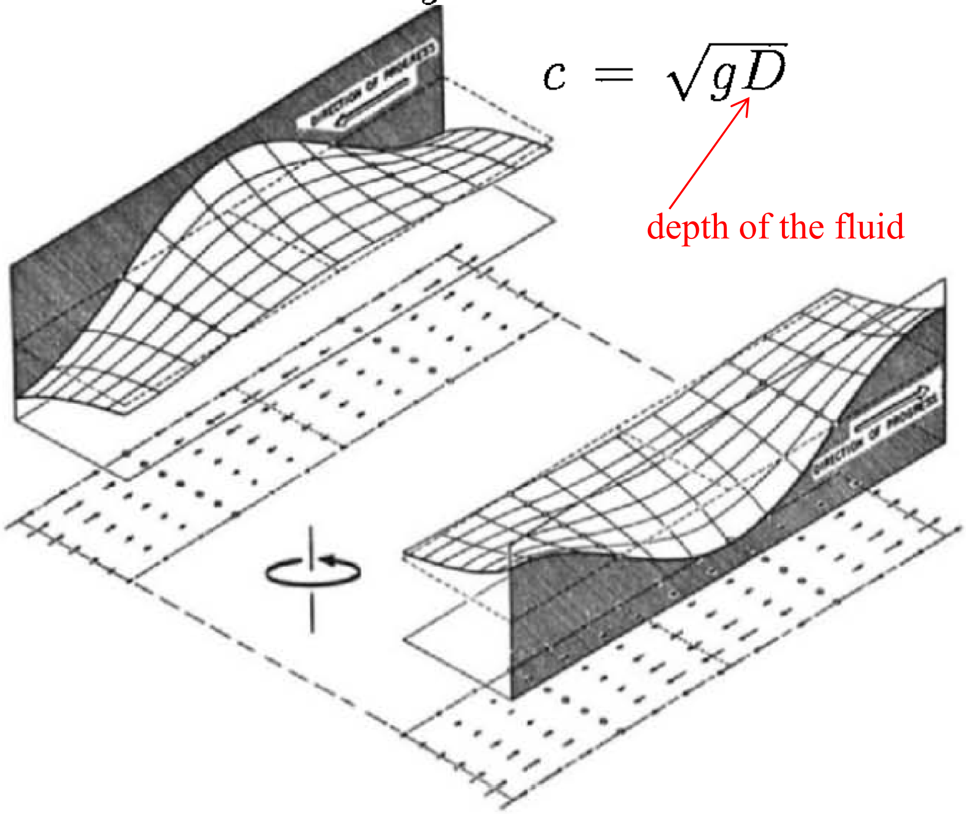


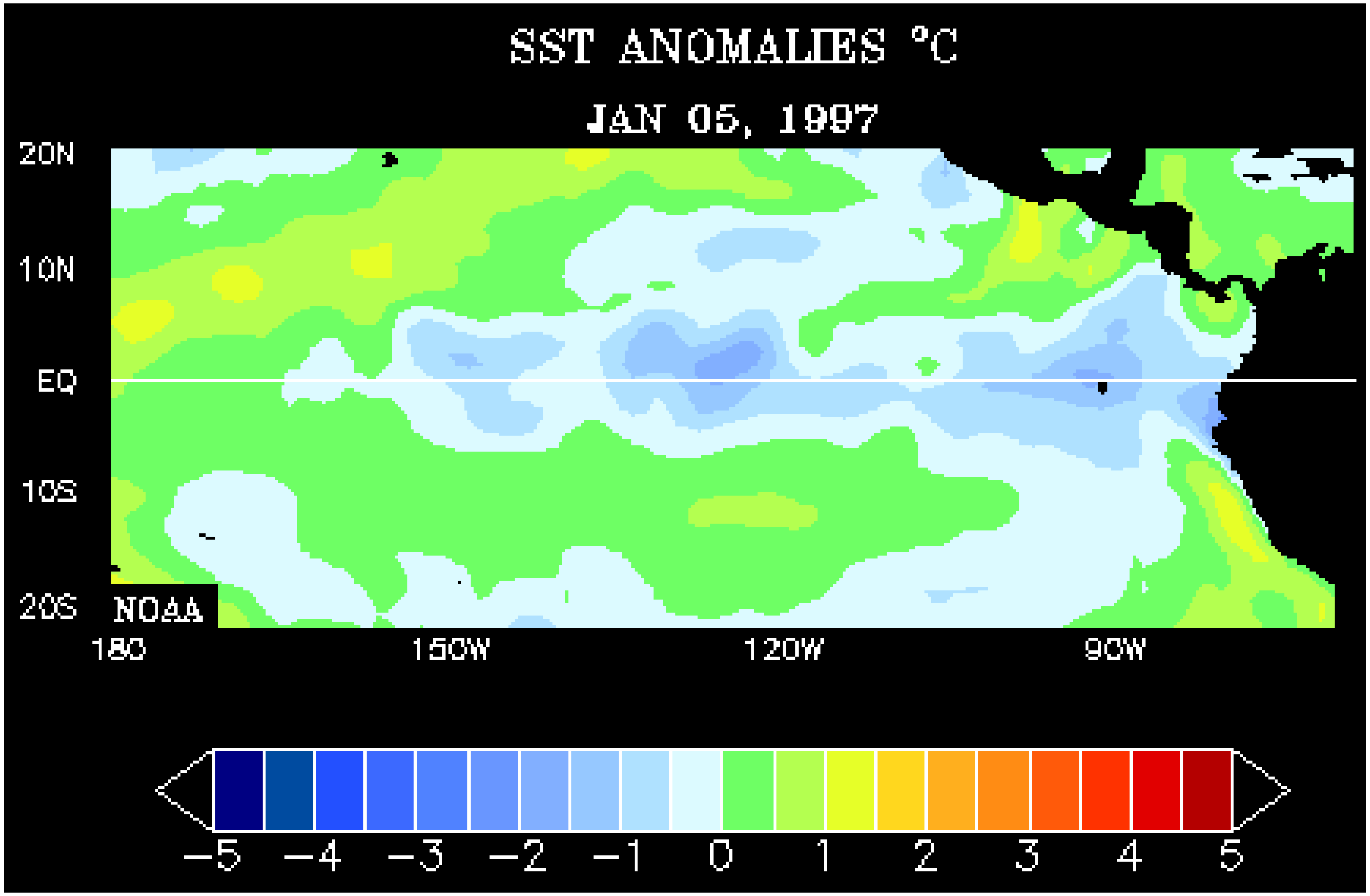
Fig. 10.3. Northern hemisphere Kelvin waves on opposite sides of a channel that is wide compared with the Rossby radius. In each vertical plane parallel to the coast, the currents (shown by arrows) are entirely within the plane and are exactly the same as those for a long gravity wave in a nonrotating channel. However, the surface elevation varies exponentially with distance from the coast in order to give a geostrophic balance. This means Kelvin waves move with the coast on their right in the northern hemisphere and on their left in the southern hemisphere. [From Mortimer (1977).]

- *Coastal Kelvin waves always propagate with the shoreline* on the right in the northern hemisphere and on the left in the southern hemisphere.
- In each vertical plane to the coast, the currents (shown by arrows) *are entirely within the plane* and are exactly the same as those for a long gravity wave in a non-rotating channel.
- However, the surface elevation varies exponentially with distance from the coast in order to give a geostrophic balance.

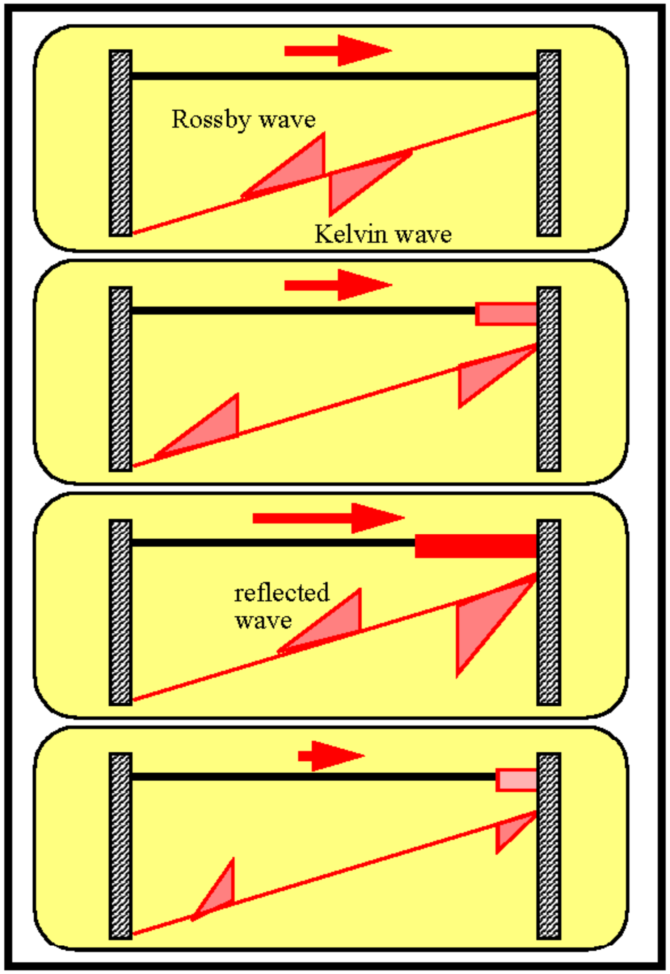
Equatorial Kelvin Waves

- The equator acts analogously to a topographic boundary for both the Northern and Southern Hemispheres, which make the equatorial Kelvin wave to behaves very similar to the coastally-trapped Kelvin wave.
- Surface equatorial Kelvin waves travel very fast, at about 200 m per second. Kelvin waves in the thermocline are however much slower, typically between 0.5 and 3.0 m per second.
- They may be detectable at the surface, as sea-level is slightly raised above regions where the thermocline is depressed and slightly depressed above regions where the thermocline is raised.
- The amplitude of the Kelvin wave is several tens of meters along the thermocline, and the length of the wave is thousands of kilometres.
- Equatorial Kelvin waves can only travel eastwards.

1997-98 El Nino

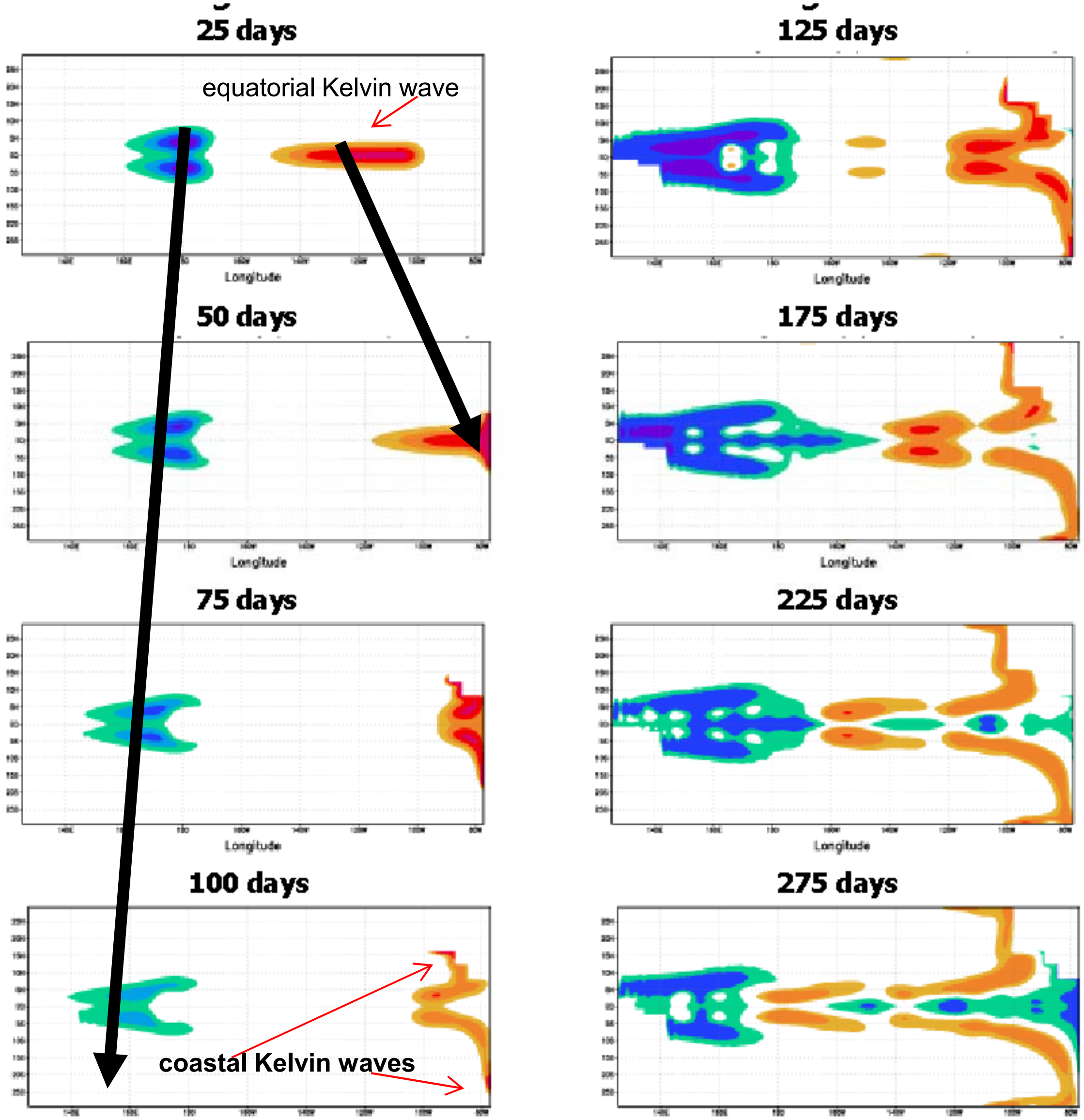


Delayed Oscillator Theory



- ❑ Wind forcing at the central Pacific: produces a downwelling Kevin wave propagating eastward and a upwelling Rossby wave propagating westward.
- ❑ wave propagation: the fast kelvin wave causes SST warming at the eastern basin, while slow Rossby wave is reflected at the western boundary.
- ❑ wave reflection: Rossby wave is reflected as a upwelling Kelvin wave and propagates back to the eastern basin to reverse the phase of the ENSO cycle.
- ❑ ENSO period: is determined by the propagation time of the waves.

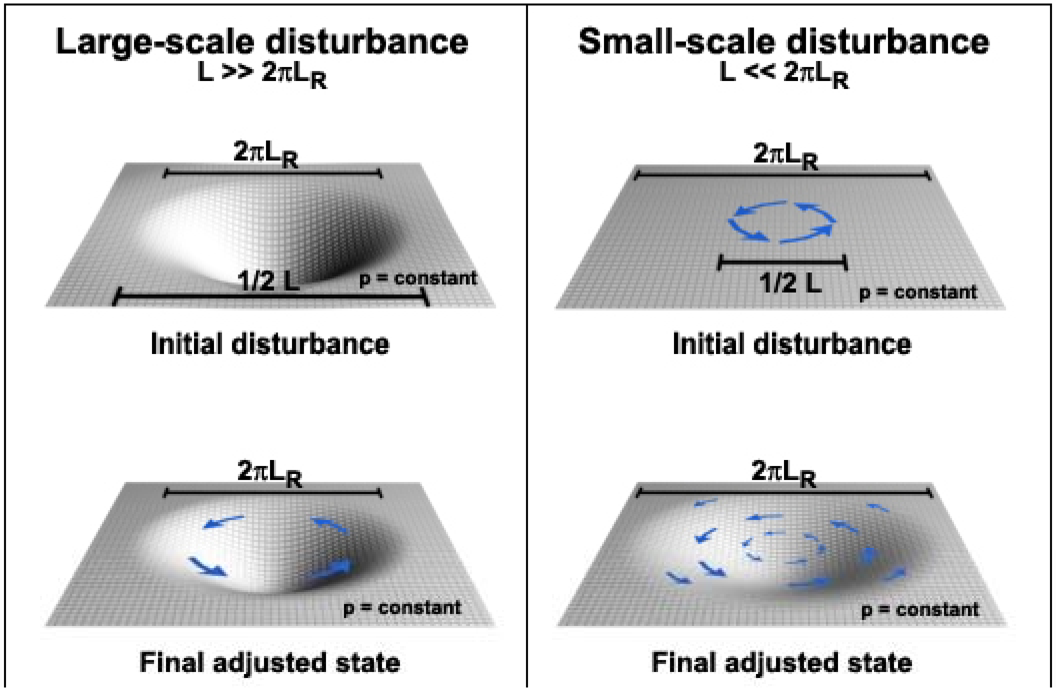
Wave Propagation and Reflection



(Figures from IRI)

- ☐ It takes Kevin wave (phase speed = 2.9 m/s) about 70 days to cross the Pacific basin (17,760km).
- ☐ It takes Rossby wave about 200 days (phase speed = 0.93 m/s) to cross the Pacific basin.

Lecture 8: Adjustment in a Rotating System



- Geostrophic Adjustment Process
- Rossby Radius of Deformation

Geostrophic Adjustments

- The atmosphere is nearly always close to geostrophic and hydrostatic balance.
- If this balance is disturbed through such processes as heating or cooling, the atmosphere adjusts itself to get back into balance. This process is called *geostrophic adjustment.*
- A key feature in the geostrophic adjustment process is that pressure and velocity fields have to adjust to each other in order to reach a geostrophic balance. When the balance is achieved, the flow at any level is along the isobars.
- We can study the geostrophic adjustment by studying the adjustment in a barotropic fluid using the shallow-water equations.
- The results can be extended to a baroclinic fluid by using the concept of equivalent depth.

Geostrophic Adjustment Problem

shallow water model

$$\frac{\partial u'}{\partial t} - f_0 v' = -g\frac{\partial h'}{\partial x}$$

$$\frac{\partial v'}{\partial t} + f_0 u' = -g\frac{\partial h'}{\partial y}$$

$$\frac{\partial h'}{\partial t} + H\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) = 0$$

$$\Longrightarrow \frac{\partial^2 h'}{\partial t^2} - c^2\left(\frac{\partial^2 h'}{\partial x^2} + \frac{\partial^2 h'}{\partial y^2}\right) + f_0 H\zeta' = 0$$

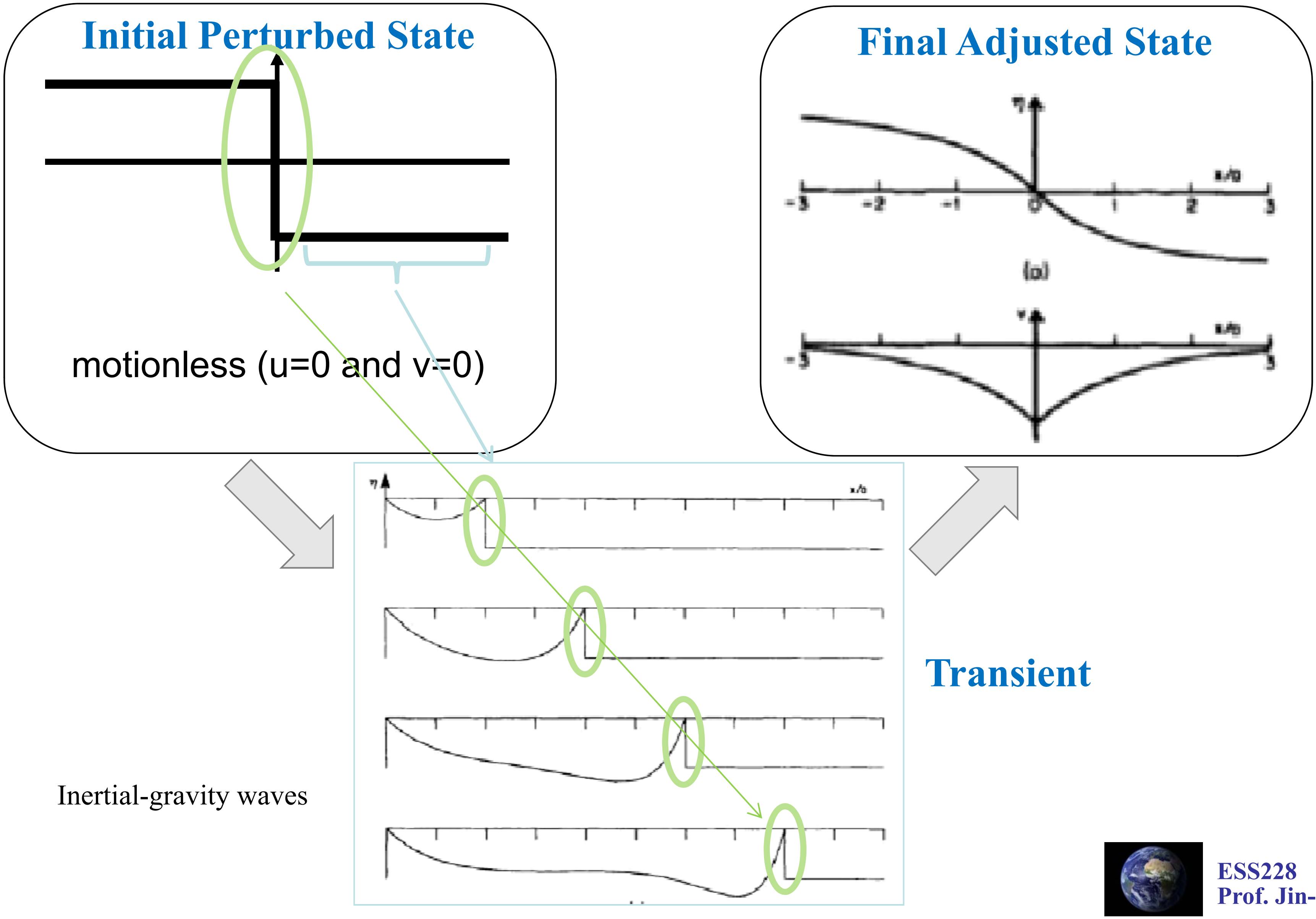
$$\frac{\partial \zeta'}{\partial t} + f_0\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) = 0 \Longrightarrow \frac{\partial \zeta'}{\partial t} - \frac{f_0}{H}\frac{\partial h'}{\partial t} = 0 \Longrightarrow \boxed{Q'(x, y, t) = \zeta'/f_0 - h'/H = \text{Const.}}$$

$$\Downarrow$$

If we know the distribution of perturbation potential vorticity (Q') at the initial time, we know for all time: $Q'(x, y, t) = Q'(x, y, 0)$

And the **final adjusted state** can be determined without solving the time-dependent problem.

An Example of Geostrophic Adjustment



Initial Perturbed State

motionless (u=0 and v=0)

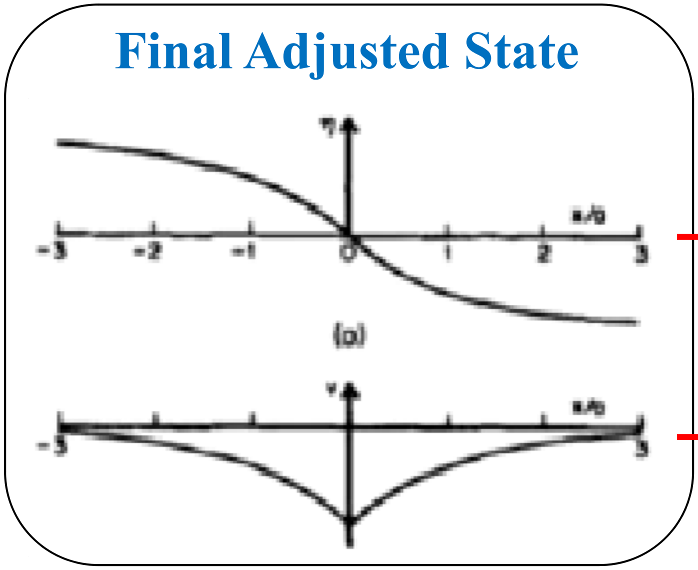
Final Adjusted State

Transient

Inertial-gravity waves

ESS228
Prof. Jin-Yi Yu

Final Adjusted State



Radius of deformation:

$$\boxed{a = c/|f| = (gH)^{1/2}/|f|}$$

$$\frac{\eta}{\eta_0} = \begin{cases} -1 + e^{-x/a} & \text{for} \quad x > 0 \\ 1 - e^{x/a} & \text{for} \quad x < 0, \end{cases}$$

$$v = -(g\eta_0/fa)\exp(-|x|/a).$$

- The steady equilibrium solution is *not one of rest, but is a geostrophic balance.*
- The equation determining this steady solution contains a length scale *a, c*alled the Rossby radius of deformation.
- The energy analysis indicates that *energy is hard to extract from a rotating* fluid. In the problem studied, there was an infinite amount of potential energy available for conversion into kinetic energy, but only a finite amount of this available energy was released. The reason was that a geostrophic equilibrium was established, and such an equilibrium retains potential energy.

Rossby Radius of Deformation

For Barotropic Flow

$$L_R \equiv \frac{(gD)^{1/2}}{f_0},$$

water depth

For Baroclinic Flow

Brunt-Vaisala frequency

$$L_R \equiv \frac{NH}{f_0},$$

equivalent depth

- In atmospheric dynamics and physical oceanography, the Rossby radius of deformation is the length scale at which *rotational effects* become as important as *buoyancy or gravity wave effects* in the evolution of the flow about some disturbance.
- “deformation”: It is the radius that the direction of the flow will be “deformed” by the Coriolis force from straight down the pressure gradient to be in parallel to the isobars.
- The size of the radius depends on the stratification (how density or potential temperature changes with height) and Coriolis parameter.
- The Rossby radius is considerably larger near the equator.

Rossby Radius and the Equilibrium State

Mass and Velocity

$$-\frac{\zeta}{f} : \frac{\eta}{H} = \kappa_H^2 a^2 : 1$$

water number

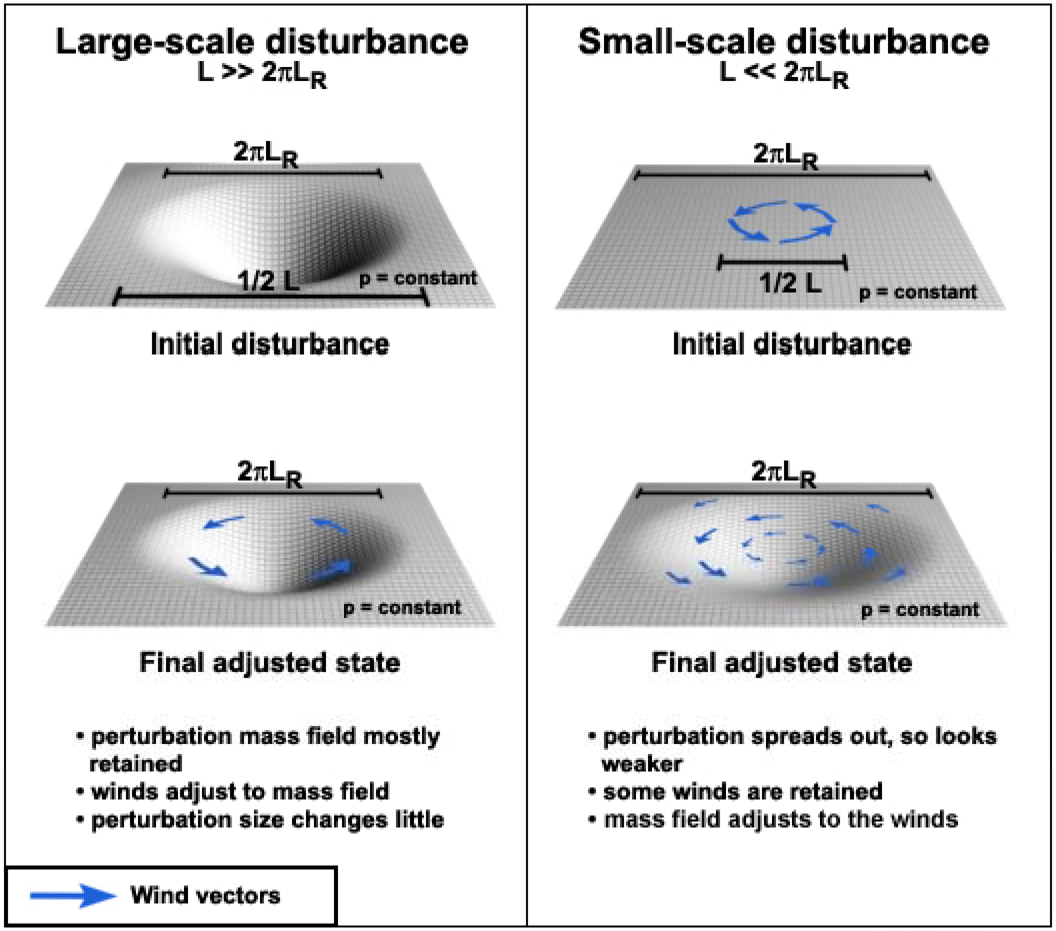
deformation radius

Energy Partition

$$\text{K.E.} : \text{P.E.} = \kappa_H^2 a^2 : 1,$$

- For large scales ($K_H a \ll 1$), the potential vorticity perturbation is mainly associated with perturbations in the mass field, and that the energy changes are in the potential and internal forms.
- For small scales ($K_H a \gg 1$) potential vorticity perturbations are associated with the velocity field, and the energy perturbation is mainly kinetic.
- At large scales ($K_H^{-1} \gg a$; *or* $K_H a \ll 1$), it is the mass field that is determined by the initial potential vorticity, and the velocity field is merely that which is in geostrophic equilibrium with the mass field. It is said, therefore, that the large-scale velocity field adjusts to be in equilibrium with the large scale mass field.
- At small scales ($K_H^{-1} \ll a$) *it is the velocity field* that is determined by the initial potential vorticity, and the mass field is merely that which is in geostrophic equilibrium with the velocity field. In this case it can be said that the mass field adjusts to be in equilibrium with the velocity field.

Rossby Radius and the Equilibrium State



- perturbation mass field mostly retained
- winds adjust to mass field
- perturbation size changes little

- perturbation spreads out, so looks weaker
- some winds are retained
- mass field adjusts to the winds

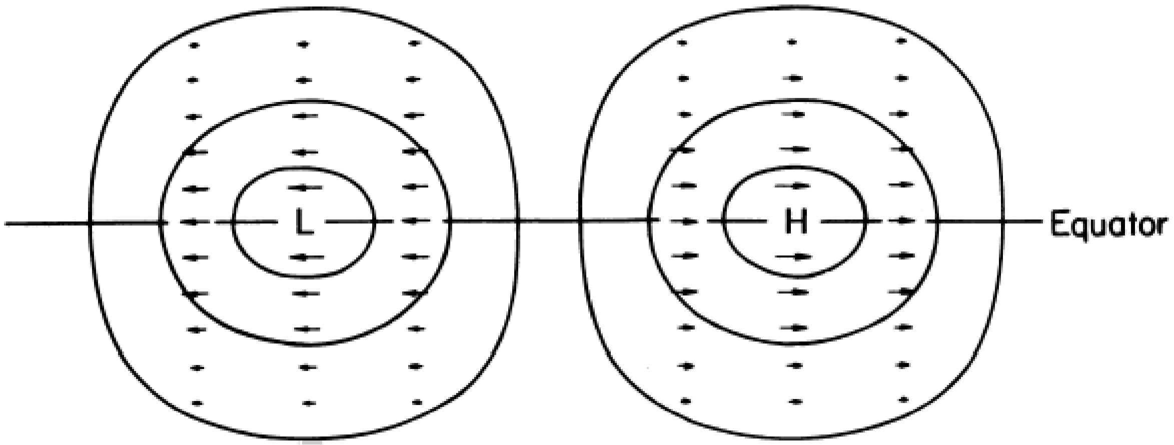
The COMET Program

- If the size of the disturbance is much larger than the Rossby radius of deformation, then the velocity field adjusts to the initial mass (height) field.
- If the size of the disturbance is much smaller than the Rossby radius of deformation, then the mass field adjusts to the initial velocity field.
- If the size of the disturbance is close to the Rossby radius of deformation, then both the velocity and mass fields undergo mutual adjustment.

What does the Geostrophic Adjustment Tell Us?

- An important feature of the response of a rotating ftuid to gravity is that it does not adjust to a state of rest, but rather to a geostrophic equilibrium.
- The Rossby adjustment problem explains why the atmosphere and ocean are nearly always close to geostrophic equilibrium, for if any force tries to upset such an equilibrium. the gravitational restoring force acts quickly to restore a near-geostrophic equilibrium.
- For deep water in the ocean, where H is 4 or 5 km. c is about 200 m/s and therefore the Rossby radius a = c/f ~ 2000 km.
- Near the continental shelves, such as for the North Sea where H=40m, the Rossby radius a = c/f ~ 200 km. Since the North Sea has larger dimensions than this, rotation has a strong effect on transient motions such as tides and surges in that ocean region.

Lecture 9: Tropical Dynamics

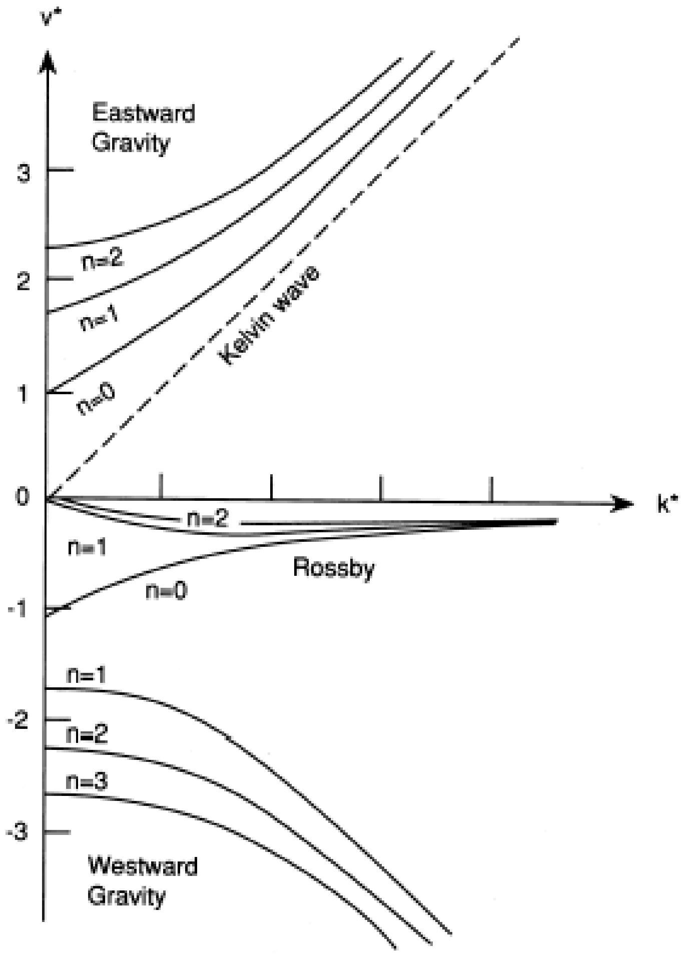


- Equatorial Beta Plane
- Equatorial Wave Theory
- Equatorial Kelvin Wave
- Adjustment under Gravity near the Eq.
- Gill Type Response

Overview

- In the Mid-latitudes, the primary energy source for synoptic-scale disturbances is the zonal available potential energy associated with the latitudinal temperature gradient; and latent heat release and radiative heating are usually secondary contributors.
- In the tropics, however, the storage of available potential energy is small due to the very small temperature gradients in the tropical atmosphere. Latent heat release appears to be the primary energy source.
- The dynamics of tropical circulations is very complicate, and there is no simple theoretical framework, analogous to quasi-geostrophic theory for the mid-latitude dynamics, that can be used to provide an overall understanding of large-scale tropical motions.

Equatorial Waves



- Equatorial waves are an important class of eastward and westward propagating disturbances in the atmosphere and in the ocean that are trapped about the equator (i.e., they decay away from the equatorial region).
- Diabatic heating by organized tropical convection can excite atmospheric equatorial waves, whereas wind stresses can excite oceanic equatorial waves.
- Atmospheric equatorial wave propagation can cause the effects of convective storms to be communicated over large longitudinal distances, thus producing remote responses to localized heat sources.

Equatorial β-Plane Approximation

- *f-plane approximation*: On a rotating sphere such as the earth, f varies with the sine of latitude; in the so-called f-plane approximation, this variation is ignored, and a value of *f* appropriate for a particular latitude is used throughout the domain.
- *β-plane approximation*: *f* is set to vary linearly in space.
- The advantage of the beta plane approximation over more accurate formulations is that it does not contribute nonlinear terms to the dynamical equations; such terms make the equations harder to solve.
- *Equatorial β-plane approximation*:

$$\left\{\begin{array}{l} cos\varphi \approx 1, \\ \sin\varphi \approx y/a. \\ f \approx \beta y \quad \text{and} \quad \beta = 2\Omega/r \approx 2.3 \times 10^{-11} \ \ \mathrm{m}^{-1}\ \mathrm{s}^{-1} \end{array}\right.$$

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Shallow-Water Equation on an Equatorial β-Plane

Linearized shallow-water equations

$$\partial u'/\partial t - \beta y v' = -\partial \Phi' \partial x$$

$$\partial v'/\partial t + \beta y u' = -\partial \Phi' \partial y$$

$$\partial \Phi'/\partial t + g h_e \left(\partial u'/\partial x + \partial v'/\partial y\right) = 0$$

Assume wave-form solutions

$$\begin{pmatrix} u' \\ v' \\ \Phi' \end{pmatrix} = \begin{bmatrix} \hat{u}(y) \\ \hat{v}(y) \\ \hat{\Phi}(y) \end{bmatrix} \exp[i(kx - vt)]$$

$$-iv\hat{u} - \beta y \hat{v} = -ik\hat{\Phi}$$

$$-iv\hat{v} + \beta y \hat{u} = -\partial\hat{\Phi}/\partial y$$

$$-iv\hat{\Phi} + g h_e \left(ik\hat{u} + \partial\hat{v}/\partial y\right) = 0$$

Only if this constant equal to an odd integer that the boundary condition (v=o at y= 0) can be satisfied.

$$\frac{\partial^2 \hat{v}}{\partial y^2} + \left[\left(\frac{v^2}{g h_e} - k^2 - \frac{k}{v}\beta\right) - \frac{\beta^2 y^2}{g h_e}\right]\hat{v} = 0$$

The index n corresponds to the number of nodes in the meridional velocity profile in the domain |y| < ∞.

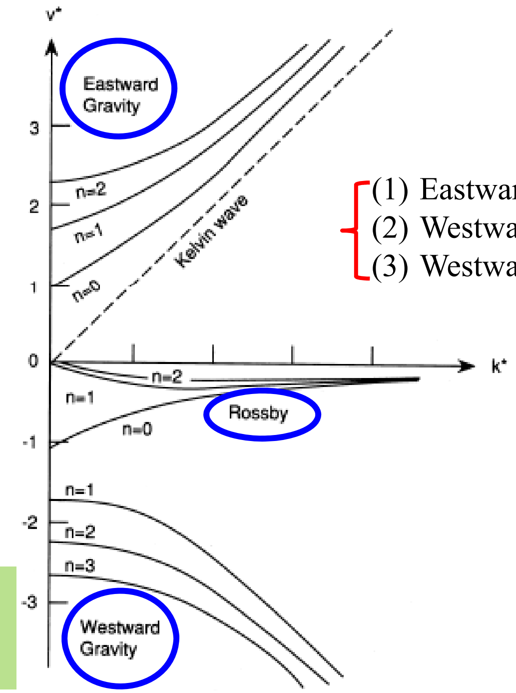
Dispersion relationship for equatorial waves

$$\frac{\sqrt{g h_e}}{\beta}\left(-\frac{k}{v}\beta - k^2 + \frac{v^2}{g h_e}\right) = 2n + 1; \quad n = 0,1,2,\ldots$$

Only these waves that satisfy the condition that the wave amplitudes decay far from the equator (where the beta-plane approximation becomes invalid.)

$$\frac{\sqrt{g h_e}}{\beta}\left(-\frac{k}{v}\beta - k^2 + \frac{v^2}{g h_e}\right) = 2n + 1; \quad n = 0,1,2,\ldots$$

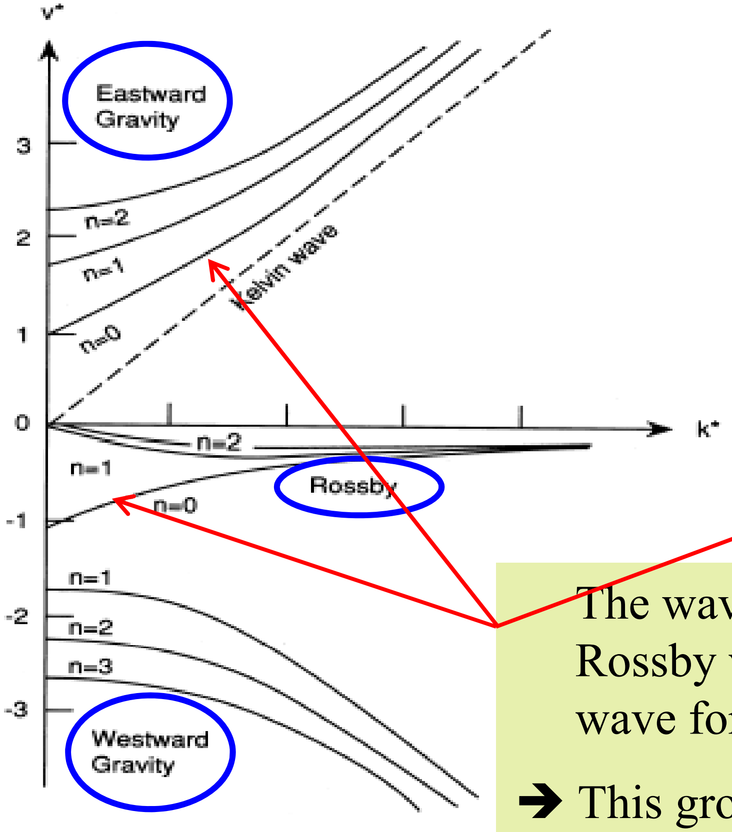
This cubic dispersion equation permit three groups of equatorially trapped waves:

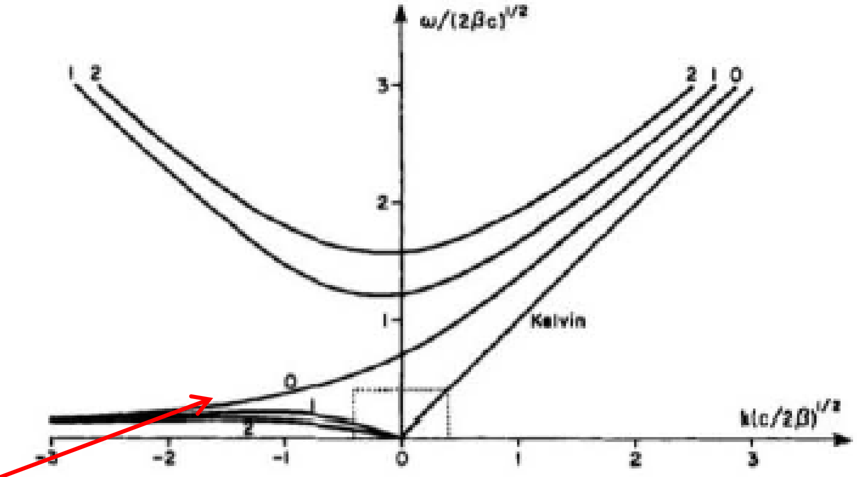


(1) Eastward-moving gravity waves
(2) Westward-moving gravity waves
(3) Westward-moving Rossby waves

Equatorial Waves with n=0

(Mixed Rossby-Gravity Waves)





The wave solution with n=0 is special, which behaves like a Rossby wave for large negative k but behaves like a gravity wave for large positive k.

➔ This group of waves is called “mixed Rossby-Gravity Waves”.

Mixed Rossby-Gravity Waves

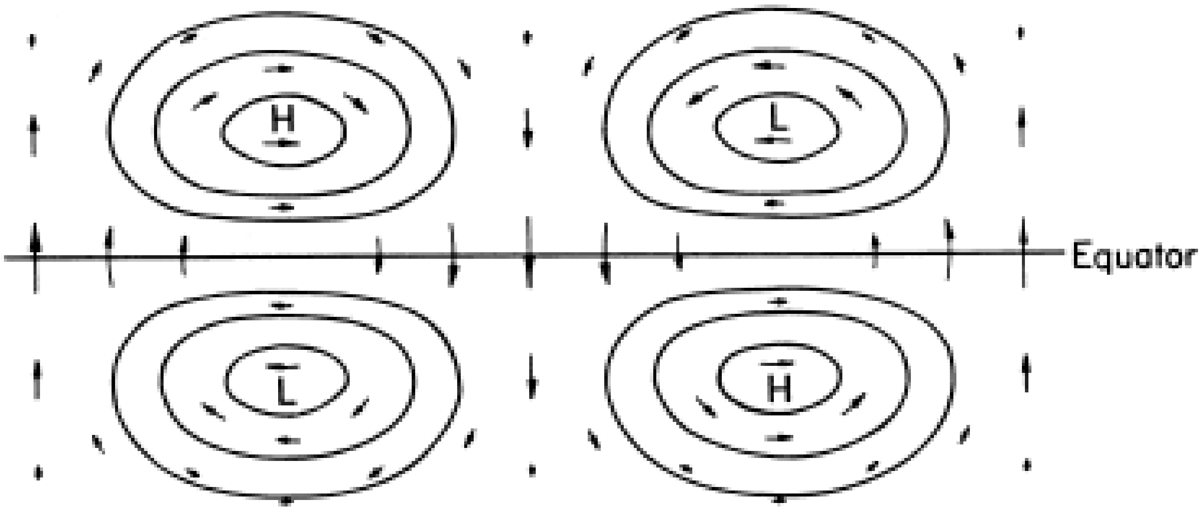
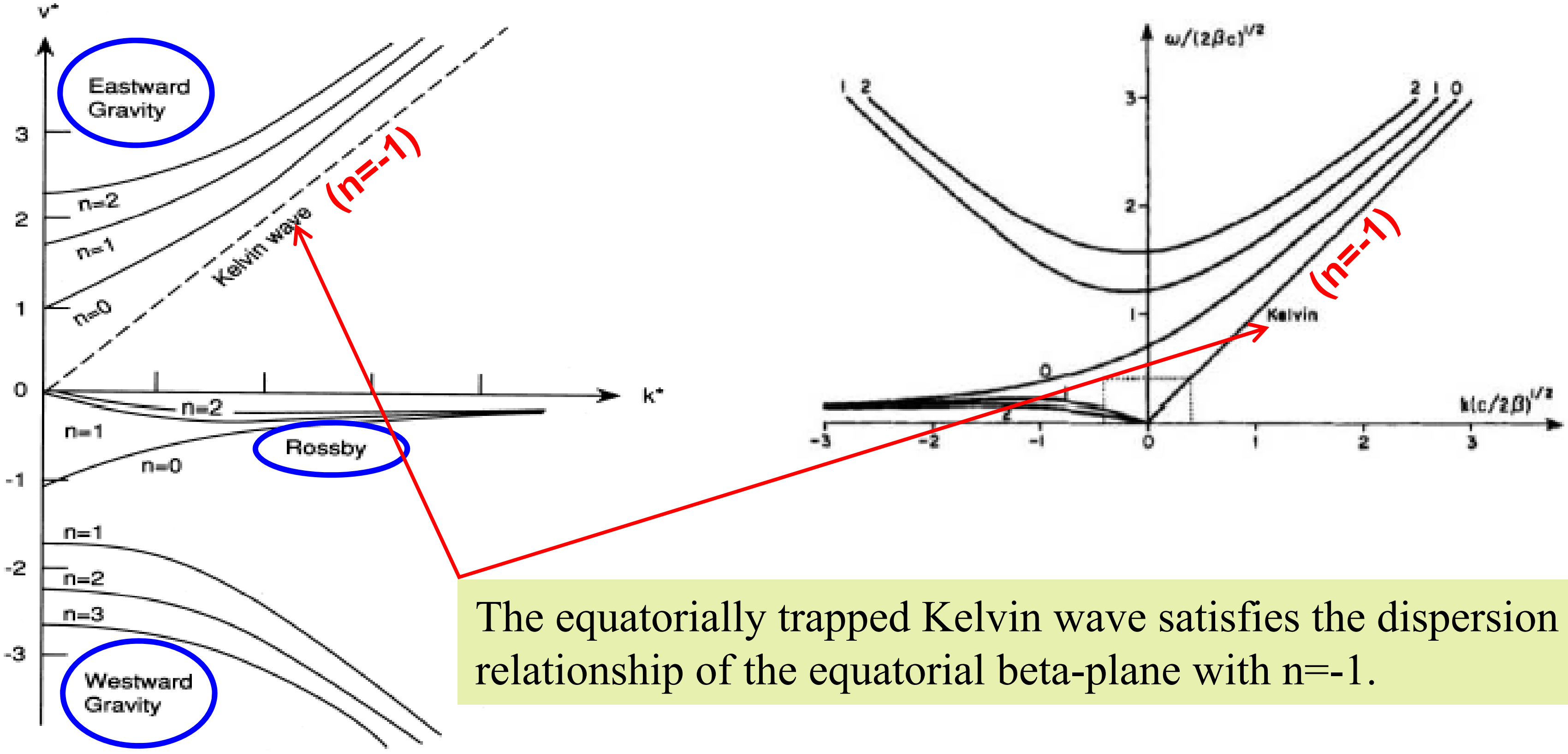


Fig. 11.13 Plan view of horizontal velocity and height perturbations associated with an equatorial Rossby–gravity wave. (Adapted from Matsuno, 1966.)

The phase velocity can be to the east or west, but the group velocity is always eastward, being a maximum for short waves with eastward group velocity (gravity waves).

Equatorial Waves with “n=-1”

(Equatorial Kelvin Waves)



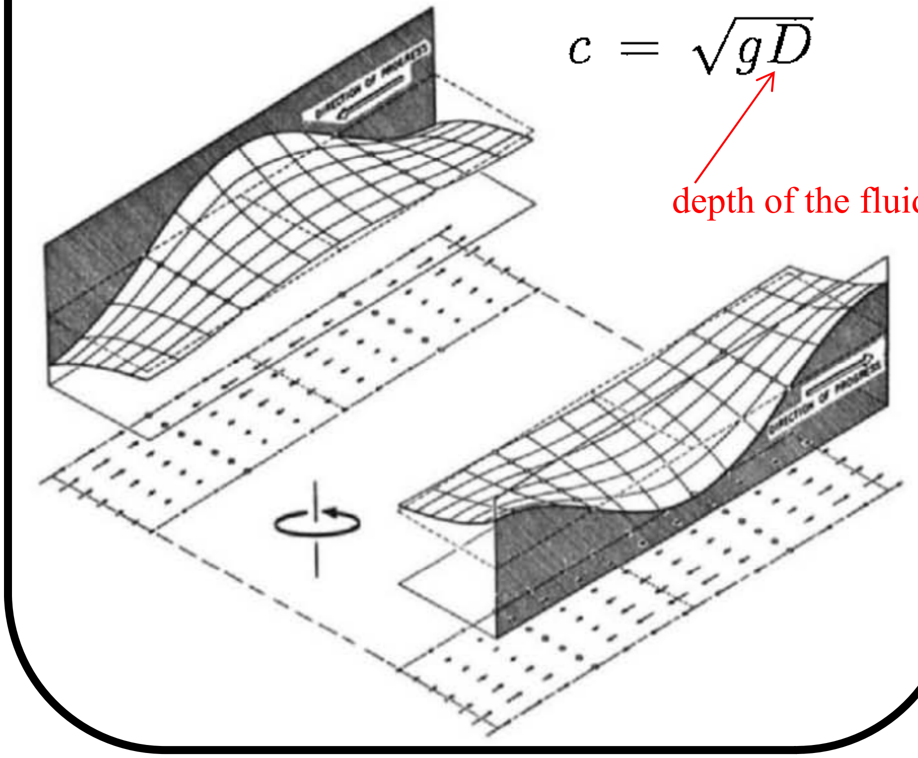
The equatorially trapped Kelvin wave satisfies the dispersion relationship of the equatorial beta-plane with n=-1.

Equatorial Kelvin Waves

Coastal Kelvin Wave

$$H = const \times \exp\left(-\frac{f}{c}y\right)$$

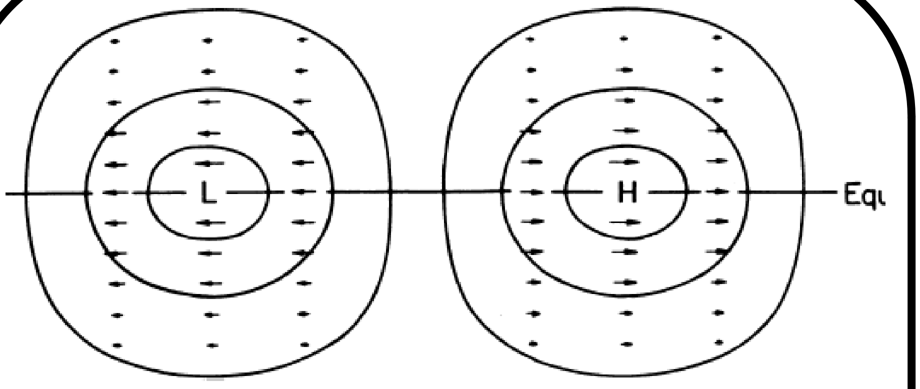
At the coast $y = 0$ is $v = 0$:



$$c = \sqrt{gD}$$

depth of the fluid

Equatorial Kelvin Wave



$$\hat{u} = u_o \exp\left(-\beta y^2/2c\right)$$

$Y_K = |2c/\beta|^{1/2}$ → e-decaying width

for a phase speed c = 30ms−1 gives Y_K ≈ 1600 km.

Equatorial Waveguide

Linearized shallow-water equations

$$\partial u'/\partial t - \beta y v' = -\partial \Phi' \partial x$$

$$\partial v'/\partial t + \beta y u' = -\partial \Phi' \partial y$$

$$\partial \Phi'/\partial t + g h_e \left(\partial u'/\partial x + \partial v'/\partial y\right) = 0$$

Assume wave-form solutions

$$\begin{pmatrix} u' \\ v' \\ \Phi' \end{pmatrix} = \begin{bmatrix} \hat{u}(y) \\ \hat{v}(y) \\ \hat{\Phi}(y) \end{bmatrix} \exp\left[i(kx - \nu t)\right]$$

$$-i\nu\hat{u} - \beta y \hat{v} = -ik\hat{\Phi}$$

$$-i\nu\hat{v} + \beta y \hat{u} = -\partial\hat{\Phi}/\partial y$$

$$-i\nu\hat{\Phi} + g h_e \left(ik\hat{u} + \partial\hat{v}/\partial y\right) = 0$$

$$\frac{\partial^2 \hat{v}}{\partial y^2} + \underbrace{\left[\left(\frac{\nu^2}{g h_e} - k^2 - \frac{k}{\nu}\beta\right) - \frac{\beta^2 y^2}{g h_e}\right]}\hat{v} = 0$$

- Wave-like solutions can result from this equation if this coefficient has a positive value.
- When y increases (i.e., away from the equator), this coefficient becomes negative (due to the $\beta^2 y^2$ term) and the wave-like solution becomes an exponential solution (i.e., the wave solution disappears).
- Equatorial waves are trapped in the tropics.
- This waveguide effect is due entirely to the variation of Coriolis parameter with latitude.

Quai-Biennial Oscillation (QBO)

(in Stratosphere)

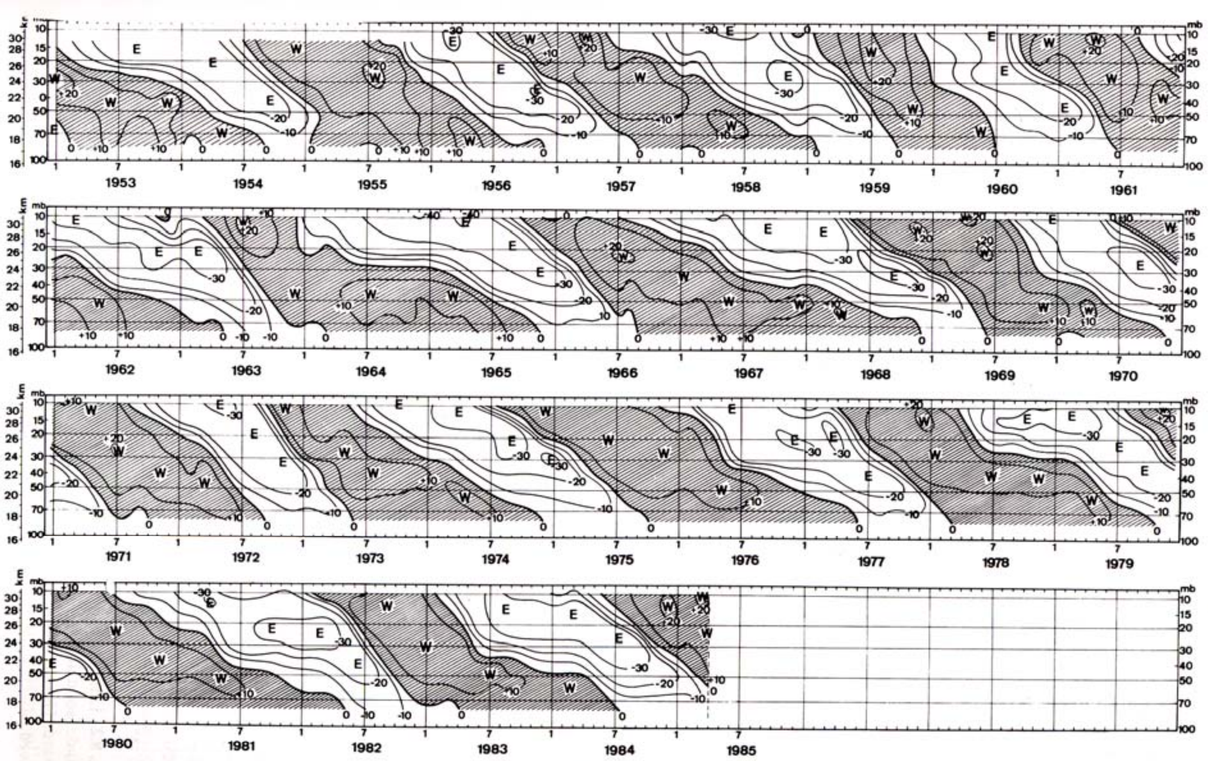


Fig. 8.1. Time–height section of monthly mean zonal winds (m s^{-1}) at equatorial stations (Jan. 1953–Aug. 1967: Canton Island, 3°S/172°W; Sept. 1967–Dec. 1975: Gan/Maldive Islands, 1°S/73°E; Jan. 1976–Apr. 1985: Singapore, 1°N/104°E). Isopleths are at 10-m s^{-1} intervals. Note the alternating downward propagating westerly (W) and easterly (E) regimes. [From Naujokat (1986), with permission.]

❑ Quasi-Biennial Oscillation: Easterly and westerly winds alternate every other years (approximately) in the lower to middle parts of the tropical stratosphere.

Why QBO?

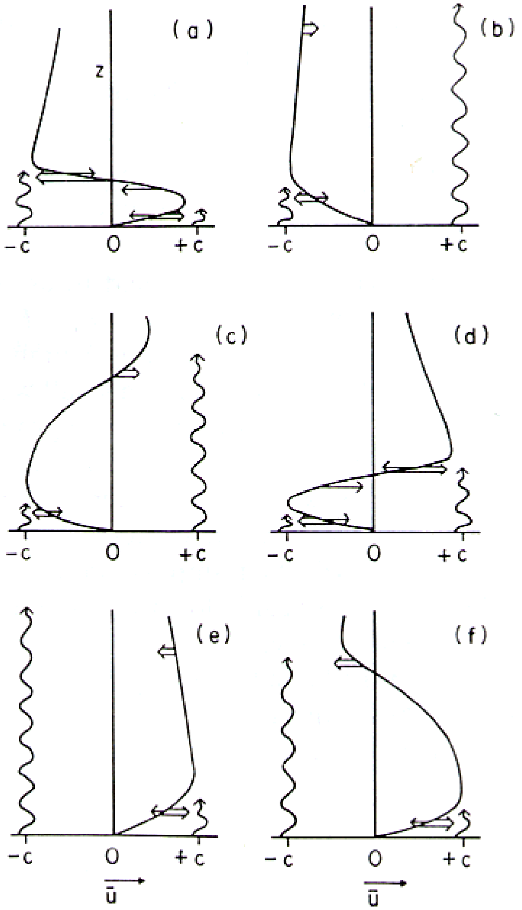


Fig. 8.7. Schematic representation of the evolution of the mean flow in Plumb's analog of the QBO. Six stages of a complete cycle are shown. Double arrows show wave-driven accelerations and single arrows show viscously driven accelerations. Wavy lines indicate relative penetration of easterly and westerly waves. See text for details. [After Plumb (1984).]

- ☐ Kevin Waves accelerates westerly.
- ☐ Mixed Rossby-Gravity Wave accelerates easterly.

- The vertically propagating equatorial Kelvin wave and mixed Rossby-gravity waves provide the zonal momentum source of QBO.
- The eastward propagating Kelvin wave provides the needed source of westerly momentum, and the westward propagating mixed Rossby-gravity wave provides easterly momentum.
- The westerly Kelvin waves tend to be damped in westerly shear zones, and the easterly mixed Rossby-gravity wave is damped in easterly shear zones.

Steady Forced Equatorial Motions

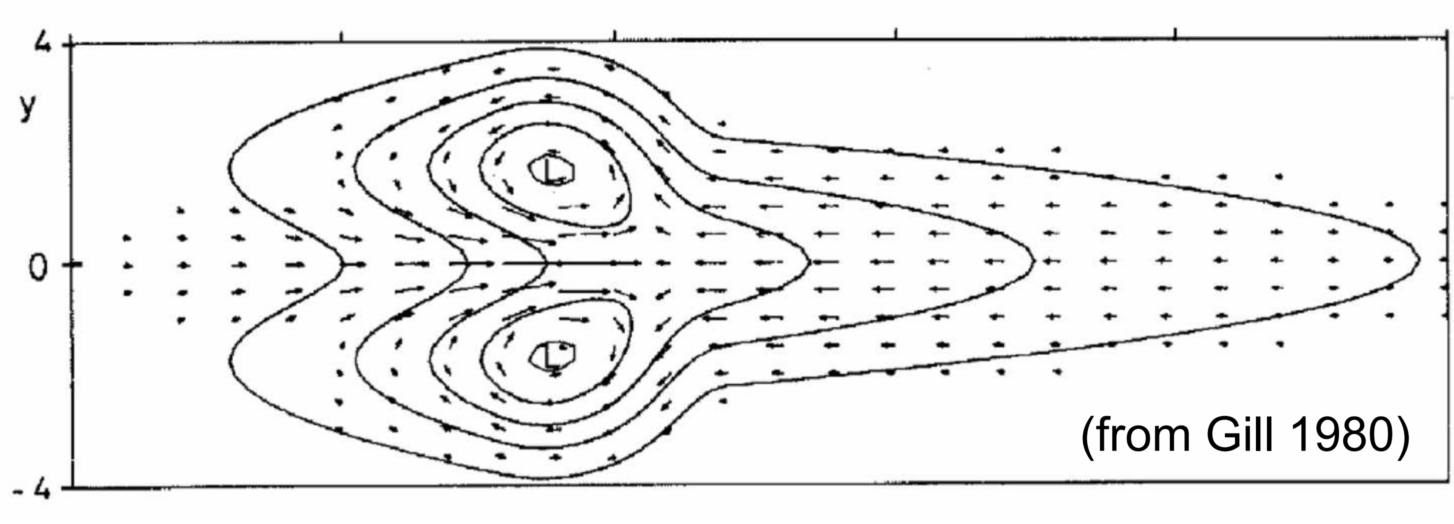
Vertically averaged equations for steady motions in the mixed layer:

$$\begin{cases} \alpha u - \beta y v + \partial \Phi / \partial x = 0 \\ \alpha v + \beta y u + \partial \Phi / \partial y = 0 \\ \alpha h + H_b (1 - \varepsilon)(\partial u / \partial x + \partial v / \partial y) = 0 \end{cases}$$

This model can be used to compute the steady surface circulation.

- The dynamics of steady circulations in a equatorial mixed layer can be approximated by a set of linear equations analogous to the equatorial wave equations , but with the time derivative terms replaced by linear damping terms.
- In the momentum equations the surface eddy stress is taken to be proportional to the mean velocity in the mixed layer.
- In the continuity equation the perturbation in the mixed layer height is proportional to the mass convergence in the layer, with a coefficient that is smaller in the presence of convection than in its absence, due to ventilation of the boundary layer by convection.

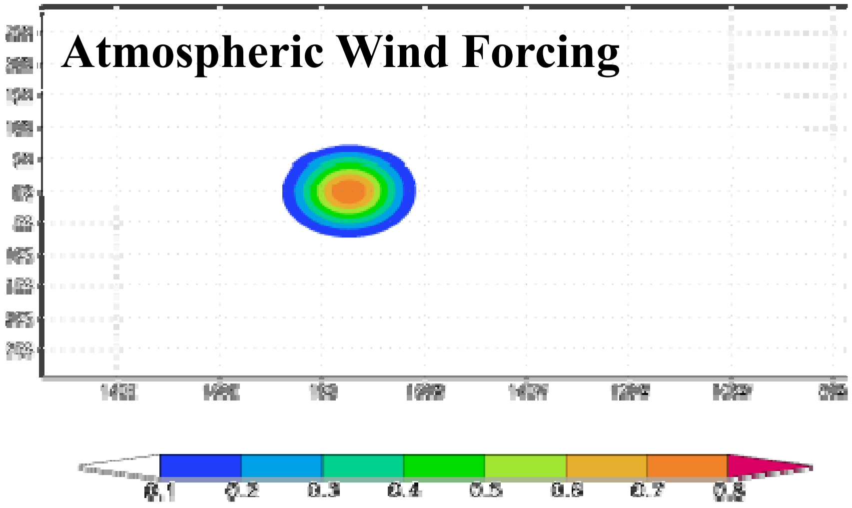
Gill’s Response to Symmetric Heating



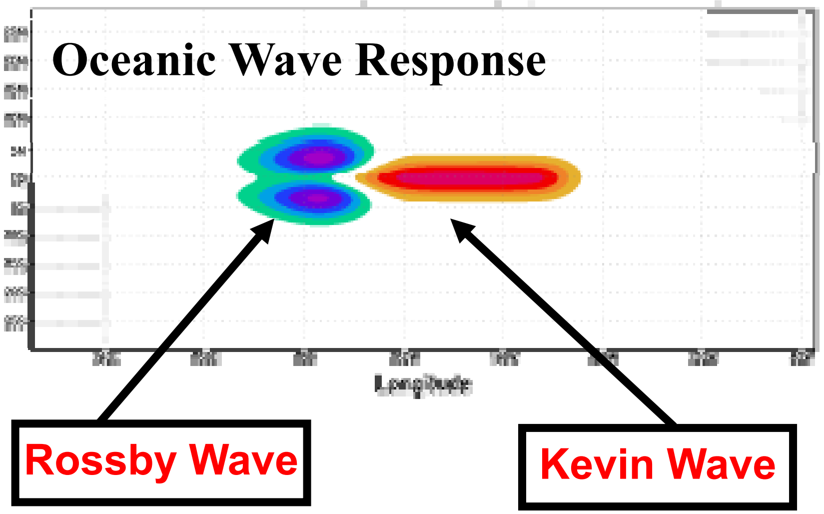
(from Gill 1980)

- This response consists of a eastward-propagating Kelvin wave to the east of the symmetric heating and a westward-propagating Rossby wave of n=1 to the west.
- The Kelvin wave low-level easterlies to the east of the heating, while the Rossby wave produces low-level westerlies to the west.
- The easterlies are trapped to the equator due to the property of the Kelvin wave.
- The n=1 Rossby wave consists of two cyclones symmetric and straddling the equator.
- The meridional scale of this response is controlled by the equatorial Rossby radius, which is related to the β-effect and the stability and is typically of the order of 1000km.

Delayed Oscillator: Wind Forcing



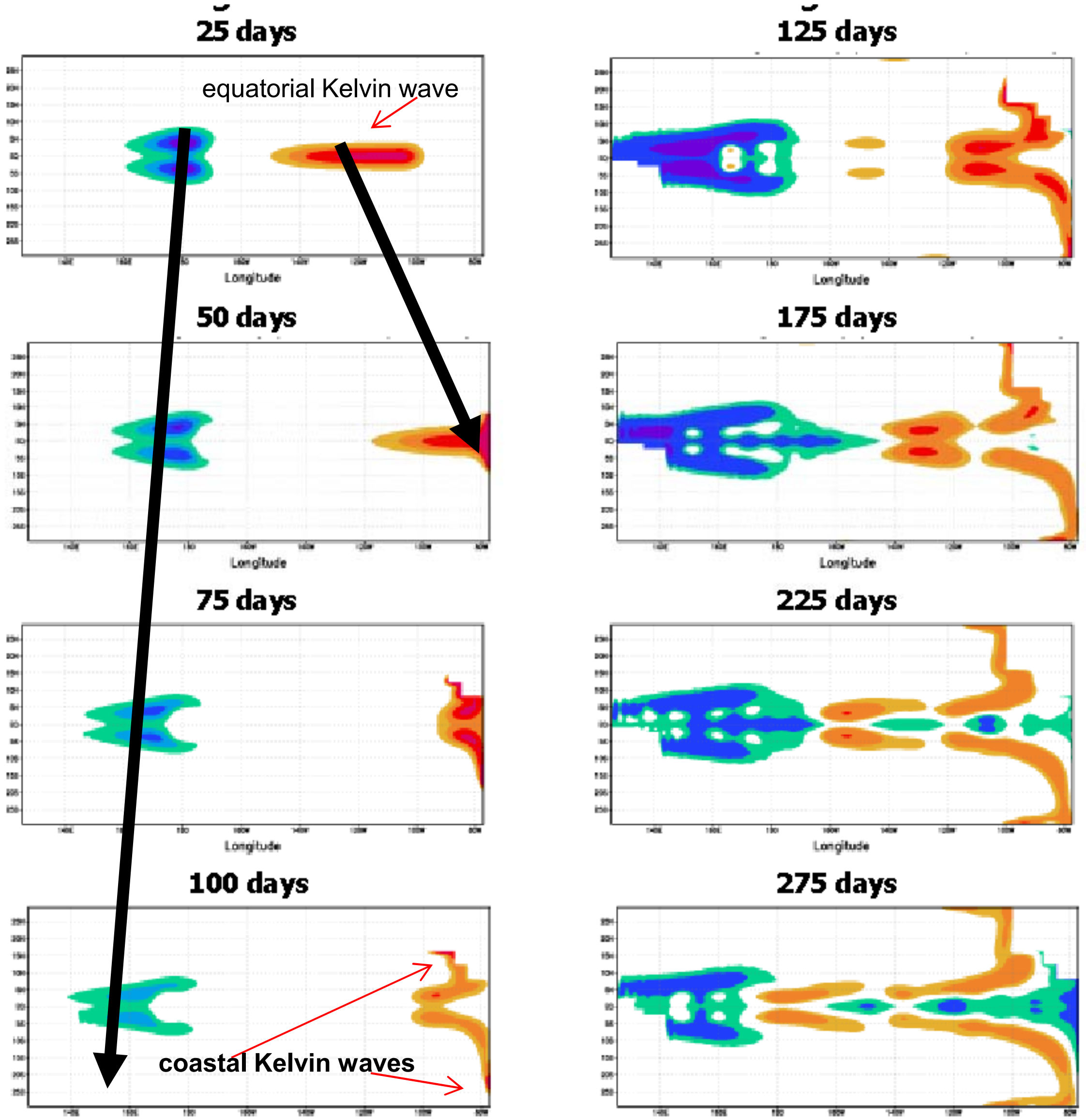
(Figures from IRI)



- The delayed oscillator suggested that oceanic Rossby and Kevin waves forced by atmospheric wind stress in the central Pacific provide the phase-transition mechanism (I.e. memory) for the ENSO cycle.

- The propagation and reflection of waves, together with local air-sea coupling, determine the period of the cycle.

Wave Propagation and Reflection



(Figures from IRI)

- It takes Kevin wave (phase speed = 2.9 m/s) about 70 days to cross the Pacific basin (17,760km).
- It takes Rossby wave about 200 days (phase speed = 0.93 m/s) to cross the Pacific basin.